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SOLUTION TECHNIQUES FOR REALISTIC PURSUIT-EVASION GAMES*

Josef Shinar

Department of Aeronautical Engineering,

Technion - Israel Institute of Technology,

Haifa, Israel

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Two solution techniques are presented and applied to solve in a closed form realistic air combat problems modelled as perfect information zero-sum differential games. Optimal missile guidance and avoidance solved as a linear differential game with bounded control. Medium range air to air interception is analysed applying the method of singular perturbations. For more complex problems the combination of the two techniques may be required.

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SOLUTION TECHNIQUES FOR

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Josef Shinar

Department of Aeronautical Engineering,

Technion - Israel Institute of Technology,

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TABLE OF CONTENTS

I. INTRODUCTION

The very nature of pursuit-evasion problems, as continous dynamics an opposing objectives, invites their formulation as two person zero-sum dif· ferential games. The study of differential games emerged from the pioneering work of Rufus Isaacs [1] in optimal pursuit and evasion applie to tactical air combat problems. It was hoped that the innovative concep of differential games (coined by Isaacs himself) would create an insight into the intriguing problems of aerial "dogfight" and eventually lead to improved tactics and better design of aircraft and weapon systems.

During more than a quarter of the century since the first RAND repc of Isaacs on pursuit games in 1951 [2], numerous investigations, deali with different aspects of this class of problems, have appeared in the literature. A considerable part of these studies, some of them reviewed other chapters of this Volume [3, 4], was aimed at the solution of air t air combat problems. Unfortunately, this extensive research effort has only had a limited impact on practical applications. This frustrating f both for scientists and military experts, can be attributed to the follo reasons:

a. In the mathematical models used in most analytical studies, im portant elements of the "real world" situation have been negle
b. Not all air combat problems satisfy the basic requirements o pursuit-evasion game formulation, such as fixed roles of the pl and diagonally opposed (zero-sum) objectives.

An obvious example for (b) is the well known "dogfight" situation between two fighter airplanes. Such an engagement, in which both pilots wish to assume an agressive role, calls for a mathematical formulation of a two-target game [5]. Moreover, in many air combat engagements, a "mutual kill," which is an equally disadvantageous outcome for both participants, is a possible outcome. Consequently, the zero-sum notion is not adequate for such problems and other concepts such as "preference-ordered gaming" [6] are required.

A careful elimination process has led to the identification of some types of air combat problems well suited to the pursuit-evasion game formulation:

a. The missile-aircraft engagement.

b. Interception of an airplane not equipped with an air to air weapon.

Both problems are described by multidimensional non-linear differential equations, which leave no hope for a closed form solution. The numerical solution of the resulting non-linear two point boundary value problem [7, 8] is not only tedious and time consuming, but may sometimes be even misleading. Computation of optimal trajectories, based on local necessary conditions, can be meaningless if unidentified singular surfaces of the game space are crossed.

Important simplifications of the mathematical model can be achieved by adopting some of the following assumptions: two-dimensional motion,

constant speeds, point-mass approximation, instantaneous response, traject linearization. However, extreme care has to be taken not to use assumptio which may eradicate salient phenomena of the original problem.

This Chapter presents two techniques for solving pursuit-evasion games of degree. Combined with a skillful mathematical modeling, it has the potential for closed form solutions of near real world air combat problems. In Section II the equations of motion of a general air combat oriented pursuit-evasion problem are given and the corresponding zero-sum perfect information differential game is formulated. In Section III the validity of several simplifying assumptions is discussed and guidelines fc appropriate mathematical modeling are suggested.

In the following sections, two methods are proposed to solve proper modeled pursuit-evasion games. In Section IV the formal solution of line differential games of terminal cost with bounded control is presented and applied to a missile-aircraft end-game. In Section V singular perturbati technique is adapted for non-linear zero-sum differential games and its effectiveness is demonstrated by the solution of a simple aircraft vs. aircraft problem. In the concluding section, the relative merits of the proposed techniques are discussed and the possibility of their combinatior is indicated.

II. PROBLEM STATEMENT

In this section a realistic air-combat oriented pursuit-evasion game is formulated.

Let R be the position vector of the Pursuer (P) in the evader's (E) coordinate system (See Fig. 1),

$$R = P - E \tag{1}$$

Then the relative motion is described by

$$\dot{R} = V_P - V_E \tag{2}$$

$$\begin{aligned} \dot{V}_P &= a_P \\ \dot{V}_E &= a_E \end{aligned} \tag{3}$$

The forces, which produce the accelerations of the vehicles, are non-linear functions of the respective position and velocity. They are also governed by a set of internal variables "C" (engine r.p.m., angle of attack, bank angle, etc.),

$$\begin{aligned} \dot{a}_P &= f_P(P_P,V_P,C_P) \\ \dot{a}_E &= f_E(P_E,V_E,C_E) \end{aligned} \tag{4}$$

each having its own dynamics expressed by

$$\begin{aligned} \dot{C}_P &= g_P(P_P,V_P,C_P,u) \\ \dot{C}_E &= g_E(P_E,V_E,C_E,v) \end{aligned} \tag{5}$$

The vectors u, v are the controls of the pursuer and the evader respectively These control variables (e.g., aerodynamic surface deflections, throttle setting, etc.) are bounded, i.e., they belong to the closed compact sets

$$u \in U \qquad v \in V \tag{6}$$

Summarizing Eqs.(2)-(5), the complete game dynamics is expressed by a non-linear autonomous vector equation of n components

$$\dot{z} = F(z,u,v) \qquad z(t_0) = z_0 \tag{7}$$

where F is continuous and differentiable with respect to its arguments and z is the state vector

$$z \triangleq (R, V_P, V_E, a_P, a_E, C_P, C_E) \qquad z \in R^n \tag{8}$$

uniquely defined in a domain $\mathcal{D}$ of R^n. In addition to the dynamics, the differential game formulation requires the determination of (i) the information structure; (ii) the criterion for game termination; (iii) the pay-off function (cost).

In the present work it is assumed that perfect information on all components of the state vector is available to both players.

The termination of the game (t_f) is determined by a closed subspace (the target set) given by

$$\psi[z(t_f)] = 0 \tag{9}$$

If the target set cannot be reached in finite time, the game terminates when the distance between the players is minimal.

$$t_f = \arg \min_{t > t_0} \| R(t) \| \tag{10}$$

The cost function, to be maximized by the evader and minimized by the pursuer, is generally written as

$$J = G[z(t_f)] + \int_{t_0}^{t_f} L(z,u,v,t)\,dt \tag{11}$$

G and L both are continuous and differentiable of their arguments. The integral term is of major importance if control penalization is included. The present investigation focuses on terminal pay-off games ($L = 0$). The natural cost in pursuit evasion games is the time of capture "t_f" defined by Eq. (9) whenever the target set can be reached. Alternatively, the pay-off can be the distance of closest approach (the miss distance) defined by Eq. (10) as

$$d(t_f) \triangleq \min_{t > t_0} \| R(t) \| \tag{12}$$

The two game formulations can thus be summarized by

$$\left.\begin{array}{ll} \dot{z} = F(z,u,v) & z(t_0) = z_0 \\ u \in U \; , \; v \in V & \\ t_f \triangleq \arg\{\psi[z(t)] = 0\} & \\ J = t_f & \end{array}\right\} \tag{13}$$

$$\left.\begin{aligned} &\dot{z} = F(z,u,v) \qquad z(t_0) = z_0 \\ &u \in U\ ,\ v \in V \\ &t_f = \arg \min_{t > t_0} \|R(t)\| \\ &J = d(t_f) \end{aligned}\right\} \tag{14}$$

The solution of a differential game is a triplet consisting of an optimal strategy pair $p^*(\cdot)$, $e^*(\cdot)$ and the optimal cost J^* (the Value of the game). The optimal strategy pair has to be selected from a set of admissible (and playable) pairs. A strategy pair $p(\cdot)$, $e(\cdot)$ is admissible if the controls

$$\begin{aligned} u(t) &= p[z(t),t] \qquad u \in U \\ v(t) &= e[z(t),t] \qquad v \in V \end{aligned} \tag{15}$$

are Lebesque measureable and generate at least one solution of the state equation (7). Furthermore, an admissible strategy pair is called playable [9] if it guarantees terminatior of the game.

The solution triplet has to satisfy the saddle point inequality

$$J(z_0,t_0,p^*,e) \leq J(z_0,t_0,p^*,e^*) \triangleq J^*(z_0,t_0) \leq J(z_0,t_0,p,e^*) \tag{16}$$

The necessary conditions to be satisfied by candidate solutions of an autonomous zero-sum differential game with terminal cost, can be stated as follows [1, 9, 10]:

Let $z^*(t)$ be an optimal trajectory and assume that $J^*(z,t)$, the Value of the game, is smooth along it; then there exists a continuous vector function $\lambda(t)$ and a Hamiltonian, defined by

$$H(z,\lambda,u,v) \triangleq \lambda^T F(z,u,v) \tag{17}$$

satisfying the adjoint equation

$$\dot{\lambda}(t) = -\left(\frac{\partial H}{\partial z}\right)_{z=z^*} \tag{18}$$

and the transversality conditions which, subject to the definition in Eq.(9), takes the form

$$\lambda(t_f) = \nu \text{ grad } \psi[z(t_f)] \qquad \nu > 0 \tag{19}$$

For cases where game termination is determined by reaching the distance of closest approach, which is also the pay off, the transversality condition is expressed by

$$\lambda(t_f) = \text{grad } d(t_f) \tag{20}$$

Moreover, the Hamiltonian also satisfies

$$\min_{u \in U} H(z^*,\lambda,u,v^*) = \max_{v \in V} H(z^*,\lambda,p^*,v) = 0 \tag{21}$$

The optimality of the candidate strategy pair, obtained from (21), has to be established by sufficiency conditions, presented in Refs. [9,10].

The necessary conditions indicate that in order to attain a candidate solution of a game, a non-linear two-point boundary value problem of the order $2n$ has to be solved.

In the original pursuit-evasion game described in this section the number of state variables (Eqs. (2)-(5)) is very large since R, V_P, V_E, a_P, a_E are all three-dimensional vectors and C_P, C_E may even have more components.

The modelling effort, discussed in the next section, is aimed at reducing the number of state variables while retaining a truthful representation of the prominent features of the original problem.

III. MODELING CONSIDERATIONS

The key to a useful solution of any complex problem is the skillful choice of the simplest possible model which preserves the salient system properties under investigation. Guidelines to this effect are suggested by the following critical discussion of some, frequently used, assumptions.

A. Simplifying Assumptions

1. Two-dimensional motion

Restricting the motion of the players within a plane, results in a reduction in the number of the state variables (at least by 5). In many cases the 2-D analysis provides an initial insight into the problem. However, some inherent characteristics of the original 3-D problem may be absent in a 2-D model. Therefore, results of 2-D solutions must be carefully examined if a

"real world" 3-D interpretation is required. This point can be well illustrated by the example of optimal missile avoidance. The first used 2-D model [11] led to discover the "bang-bang" nature of the optimal maneuver, but only a later 3-D analysis [12] could define its optimal direction, which is perpendicular to the plane of collision.

2. Point-mass approximation

With this approximation vehicle dimensions and rotational degrees of freedom are disregarded. The vehicle is represented by its center of gravity. The complex non-linear dynamics of the vectors C_P, C_E in (5) can be replaced by linear differential equations and often represented only by first order time constants as:

$$\begin{aligned} \tau_P \dot{C}_P + C_P &= u \\ \tau_E \dot{C}_E + C_E &= v \end{aligned} \tag{22}$$

The disregard of vehicle dimensions has, however, a serious limitation. If the distance of closest approach is of the same order of magnitude as the size of the airplane involved, the very concept of closest approach defined by (12) may loose its meaning. However, for a case resulting in very small or very large miss distances, the point mass approximation is a useful and justifiable assumption.

3. Instantaneous control response

This frequently used assumption disregards the time lag in the control inputs by setting $\tau_P = \tau_E = 0$ in Eq.(22). This assumption leads to the conclusion that, for pursuer's speed and maneuver advantages : $V_P > V_E$, $a_P \geq a_E$, a "point capture" would be possible [13, 14]. This conclusion is disapproved by the more complete model. Thus, in problems where miss distance calculation is important, the assumption of "instantaneous response" is inadequate.

4. Constant speed

This assumption rarely represents physical reality. It can, however, be justified in problems of short duration, where the effect of velocity change is negligible and for vehicles in which the longitudinal component of the acceleration is much smaller than the lateral ones. In such cases constant speed models give a fairly good description of the main phenomena.

5. Trajectory linearization

In some pursuit-evasion problems there exists a reference trajectory allowing linearization of the originally non-linear kinematics. Collision course (see Fig. 2) is an example for such situation. Trajectory linearization is justified only if the total direction change during the engagement is not too important.

A valid trajectory linearization combined with the constant speed assumption allows to describe the relative motion by a set of linear differential equations. Since in this case the velocity components along the reference

trajectory are almost constant, changes in this direction can be expressed as a function of the time, resulting in a further reduction in dimensionality. Moreover, in such problems the capture time "t_f" can be determined.

B. Model Formulation

Any deterministic pursuit-evasion process can be divided into three phases: the initial "acquisition" phase, the main "pursuit" phase and the "end-game." In the main pursuit phase the distance of separation between the players is reduced and the state of the game approaches the target set. If the initial conditions of the engagement are unfavourable to such "pure pursuit", the "acquisition" phase becomes important. This phase is characterized by significant directional changes of the trajectories. As the game nears its termination, the attention of the "players" is focused on the conditions imposed by the terminal constraints. Consequently, the optimal strategies of the "end-game" can be very different from the ones used in other phases of the game.

These observations indicate that the best mathematical model is not necessarily the same for all phases of a pursuit-evasion game. As examples, let us examine the two air combat problems, well suited for zero-sum differential game formulation, indicated in the Introduction.

1. Missile vs. Aircraft Game

The majority of such engagements (excluding the type of "dogfight missiles," to be mentioned later) can be characterized by:

a. The pursuer has a definite advantage both in speed, $V_P > V_E$ and maneuverability, $a_P > a_E$.

b. The launching platform (either airborne or ground based) provides generally favourable initial conditions for the pursuit.

c. The outcome of the engagement can be measured by the "miss distance" (distance of closest approach), thus termination of the game is guaranteed.

In such engagements, the emphasis is obviously on the "end-game" and as a consequence of (b) the acquisition phase can be neglected. The mathematica model of this problem can be based on trajectory linearization as well as on constant speed, point mass approximations, but cannot assume instantaneous control response of the pursuer. This mathematical model, both in a 2-D or a 3-D version, yields linear time dependent differential equations of motion. Let us remark, however, that the validity of trajectory linearization and the assumption neglecting vehicle size has to be verified "a posteriori."

For future dogfight missiles of the ASRAAM type, the acquisition phase is of major importance. To describe this phase, a non-linear model must be used.

2. Aircraft vs. Aircraft Interception Game

In this engagement the lack of air to air weapon forces one of the planes to assume the role of the evader. The pursuing fighter may or may not have speed or maneuverability advantage relative to his opponent. Initial conditio

may or may not be favorable for interception. As a consequence, in this game the acquisition phase is of major importance. The interception will be successful if the pursuer can reach the evader at a distance determined by the "firing envelope" of its weapon within a finite time interval. If termination, as defined above, is possible, the natural cost function of this game is the time of capture.

Since the maximum firing range of modern air to air missiles largely exceeds the radius of turn of combat airplanes, the "end-game" phase in such engagements is hardly noticed.

In the aircraft interception game there is no requirement for accurate miss distance calculation. Consequently, the assumptions of instantaneous control response and point mass approximation can be adopted. Since the very nature of the acquisition phase does not allow trajectory linearization, game dynamics remains non-linear. The validity of constant speed and 2-D models strongly depends on problem parameters and has to be examined separately. These two assumptions seem to be tied together. In previous works [15,16], it has been shown that the optimal pursuit evasion game of constant speed vehicles is confined to a plane. This result may not be true for variable speed airplanes even if the initial conditions are two-dimensional.

C. Selection of Solution Techniques

Pursuit-evasion games of valid linear mathematical models can be analysed by the powerful methods of linear differential game theory. Though attention in the past has been focused on linear games with quadratic pay-off functions and unbounded controls [17, 18, 19, 20], examples of terminal cost linear games with hard bounded control were also solved [21, 22].

In Section IV such latter version is applied to solve the missile vs. aircraft end-game with realistic dynamics. In the sequel, implementation of the results for missile guidance as well as for missile avoidance are discussed and the validity of the linear model is examined.

For problems where trajectory linearization cannot be justified a non-linear two-point boundary value problem remains to be solved. Exact solutions in closed form exist only for problems with very low dimension [1, 23, 24] using oversimplified mathematical models. For a practical application, however, an approximate solution of a near real world model seems much more attractive. In recent years several non-linear two point boundary value problems originating in optimal control, including problems of aircraft performance optimization, have been solved using the approximation technique of singular perturbations (SPT) [25, 26, 27, 28]. Linear differential games of high dimensions were also treated by the same method [29, 30, 31]. In a recent study [32], it was proposed to apply the method of singular perturbations to non-linear differential games.

In Section V the basic notions and principles of SPT are outlined and the application for non-linear pursuit-evasion games is discussed. The

merits of the proposed approximation technique is demonstrated by a simple (2-D, constant speed) example. The method of SPT has, however, straightforward extension for more realistic (variable speed, 3-D) models.

IV. LINEAR DIFFERENTIAL GAMES WITH BOUNDED CONTROLS

Linear differential games (LDG) have been extensively investigated in the last 15 years on both sides of the iron curtain [17-20, 34-38]. However, the potential of LDG technique to solve realistic pursuit problems (i.e., games of terminal cost with bounded controls) was only recently realized [21, 22, 39-41]. In order to demonstrate its effectiveness as an analytical tool, the LDG technique will be applied in this section to solve the missile vs. aircraft engagement described in III.B.1. The solution has a clear geometric interpretation which enables to discuss the implementation of the optimal strategies as well as the validity of the linear model.

A. Formulation of the Missile vs. Aircraft Game

Based on the description of such an engagement given in the previous section (III.B.1) the following set of assumptions is adopted:

1) Both pursuer and evader are considered as point-mass vehicles.
2) The speed of each vehicle is constant, the pursuer being the faster ($V_P/V_E > 1$).
3) The relative motion is three-dimensional (See Fig. 3).

4) Gravity, having no effect on the relative trajectory, is neglected.

5) The initial conditions of the pursuit are near to a collosion course (See Fig. 2).

6) The relative trajectory can be linearized around the initial line of sight vector.

7) The performance index of the problem is the miss distance (distance of closest approach).

8) There exists perfect (complete and instantaneous) information on the state variables and the parameters of the problem.

9) The lateral acceleration commands of both vehicles are bounded by circular vectograms perpendicular to the respective velocity vectors ($a_P/a_E > 1$). This assumption will be slightly modified in the course of the solution.

10) The pursuer's response to its acceleration command is approximated by single time constant τ_P .

11) Evader dynamics can be approximated by a first order time constant τ_E .

Assumptions 2, 5, and 6 lead to a set of linear differential equations. Moreover, as a consequence of the linearization, the relative motion in the line of sight direction (the X axis) is of constant speed and the duration of the game t_f is determined.

The state vector of this problem has eight components

$$z \triangleq \text{col}\ \{Y,\dot{Y},\ddot{Y}_P,\ddot{Y}_E \ \vdots\ Z,\dot{Z},\ddot{Z}_P,\ddot{Z}_E\} \tag{23}$$

where

$$\begin{aligned} Y &= Y_P - Y_E \\ Z &= Z_P - Z_E \end{aligned} \tag{24}$$

The dynamics to be considered is perpendicular to the line of sight. In this coordinate system the circular vectorgram, perpendicular to the respective velocity vectors (see Ass. 9), becomes eliptic as depicted in Fig. 4. Accordingly, the admissible control sets u and v have the form

$$\begin{aligned} u &= \{u : u^T R u \leq a_P^2\} \\ v &= \{v : v^T S v \leq a_E^2\} \end{aligned} \tag{25}$$

with

$$R = \begin{bmatrix} 1/\cos^2 \chi_P(0) & 0 \\ 0 & 1 \end{bmatrix}$$

and (26)

$$S = \begin{bmatrix} 1/\cos^2 \chi_E(0) & 0 \\ 0 & 1 \end{bmatrix}$$

The game dynamics can be now described by

$$\dot{z} = Az + Bu + Cv \qquad z(0) = z_0 \tag{27}$$
$$u \in U \;,\; v \in V$$

with

$$A = \left[\begin{array}{c:c} A_1 & 0 \\ \hdashline 0 & A_1 \end{array}\right] ; \qquad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_P & 0 \\ 0 & 0 & 0 & -1/\tau_E \end{bmatrix}$$

$$B^T = \frac{1}{\tau_P} \left[\begin{array}{cccc:cccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$C^T = \frac{1}{\tau_E} \left[\begin{array}{cccc:cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The pay-off of the game is defined by (see Ass. 7)

$$J = \| D\, z(t_f) \| \tag{28}$$

with

$$D = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

It is required to find among all admissible strategies $\{p(\cdot), e(\cdot)\}$,– such that $u(t) = p[z(t),t]$, and $v(t) = e[z(t)t]$, – an optimal pair $\{p^*(\cdot), e^*(\cdot)\}$ satisfying the saddle point inequality

$$J(z,t,p^*,e) \leq J(z,t,p^*,e^*) \stackrel{\Delta}{=} J^*(z,t) \leq J(z,t,p,e^*) \tag{29}$$

Equations (27) and (28) can be simplified by using the following transformation:

$$y(t) = D\,\phi(t_f,t)z$$

$$\mathcal{B}(t) = D\,\Phi(t_f,t)B \qquad (30)$$

$$\mathcal{C}(t) = D\,\Phi(t_f,t)C$$

where $\phi(t_f,t)$ is the transition matrix of the linear system $\dot{z} = Az$ satisfying

$$\frac{d}{dt}\Phi(t_f,t) = -\Phi(t_f,t)A \qquad (31)$$

By Eq. (30) and (31) the original formulation of Eq.(27) and (28) becomes

$$\dot{y} = \mathcal{B}(t)u + \mathcal{C}(t)v \qquad y(0) = y_0 \qquad (32)$$
$$u \in U \quad v \in V$$

and

$$J = \|y(t_f)\| \qquad (33)$$

The new 2-D variable y can be interpreted as the vector of the "predicted miss distance." In this particular problem

$$\Phi(t_f,t)=\Phi(t_f-t,0)\stackrel{\Delta}{=}\phi(\theta)= \left[\begin{array}{c:c} \Phi_1(\theta) & 0 \\ \hdashline 0 & \Phi_1(\theta) \end{array}\right] \qquad (34)$$

"θ" being the normalized "time to go," defined as

$$\theta = \frac{t_f - t}{\tau_P} \tag{35}$$

and

$$\Phi_1(\theta) = \begin{bmatrix} 1 & \tau_P\theta & \tau_P^2\,\psi_P(\theta) & -\tau_E^2\,\psi_E(\theta) \\ 0 & 1 & \tau_P(1-\exp[-\theta]) & -\tau_E(1-\exp[-(\tau_P/\tau_E)\theta]) \\ 0 & 0 & \exp[-\theta] & 0 \\ 0 & 0 & 0 & \exp[-(\tau_P/\tau_E)\theta] \end{bmatrix} \tag{36}$$

the functions $\psi_P(\theta)$ and $\psi_P(\theta)$ are given by

$$\begin{aligned} \psi_P(\theta) &\triangleq \theta + \exp[-\theta] - 1 > 0\ ,\ \forall\ \theta > 0 \\ \psi_E(\theta) &\triangleq (\tau_P/\tau_E)\theta + \exp[-(\tau_P/\tau_E)\theta] - 1 > 0\ ,\ \forall\ \theta > 0 \end{aligned} \tag{37}$$

According to Eqs.(29) and (36) the components of the predicted miss distance are

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Y + \tau_P\theta\,\dot{Y} + \tau_P^2\,\psi_P(\theta)\,\ddot{Y}_P - \tau_E^2\,\psi_E(\theta)\,\ddot{Y}_E \\ Z + \tau_P\theta\,\dot{Z} + \tau_P^2\,\psi_P(\theta)\,\ddot{Z}_P - \tau_E^2\,\psi_E(\theta)\,\ddot{Z}_E \end{pmatrix} \tag{38}$$

and the 2x2 matrices $\mathcal{B}(t)$, $\mathcal{C}(t)$ have the form

$$\begin{aligned} \mathcal{B}(t) &= \tau_P\,\psi_P(\theta)\,I_2 \\ \mathcal{C}(t) &= -\,\tau_E\,\psi_E(\theta)\,I_2 \end{aligned} \tag{39}$$

B. Solution of the Game

The Hamiltonian of the transformed game is

$$H(y,\lambda,t) = \lambda^T \dot{y} = \lambda^T [B(t)u + C(t)v] \tag{40}$$

where $\lambda(t)$ has to satisfy

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial y} = 0 \tag{41}$$

$$\lambda(t_f) = \text{grad } J = \frac{y(t_f)}{\| y(t_f) \|} \tag{42}$$

Assuming that $\lambda(t)$ is continuous (this hypothesis must be verified) Eqs. (41) and (42) yield

$$\lambda(t) = \frac{y(t_f)}{\| y(t_f) \|} \triangleq \overset{*}{\xi} \tag{43}$$

$\overset{*}{\xi}$ being a constant unit vector along each optimal trajectory.

The candidate optimal strategy pair can be now determined by

$$\begin{aligned} &\min_{u \in U} \; \overset{*}{\xi}{}^T B(t) u = \overset{*}{\xi}{}^T B(t) p^* \\ &\max_{v \in V} \; \overset{*}{\xi}{}^T C(t) v = \overset{*}{\xi}{}^T C(t) e^* \end{aligned} \tag{44}$$

yielding

$$p^*(y,t) = -a_P M_P^T \frac{M_P \overset{*}{\xi}}{\| M_P \overset{*}{\xi} \|}$$

$$e^*(y,t) = -a_E M_E^T \frac{M_E \overset{*}{\xi}}{\| M_E \overset{*}{\xi} \|} \tag{45}$$

with

$$M_P = \begin{bmatrix} \cos \chi_P(0) & 0 \\ 0 & 1 \end{bmatrix} ; \quad M_E = \begin{bmatrix} \cos \chi_E(0) & 0 \\ 0 & 1 \end{bmatrix} \tag{46}$$

Since $\overset{*}{\xi}$ is a constant unit vector, trajectory equations can be directly integrated leading to determine the cptimal cost J^* (the Value) as well as the direction of the vector $\overset{*}{\xi}$ by

$$J^*(y,\vartheta) = \sup_{\|\xi\|=1} \left\{ \xi^T y - \alpha(\theta) \| M_P \xi \| + \beta(\theta) \| M_E \xi \| \right\} \tag{47}$$

with

$$\alpha(\theta) = a_P \tau_P^2 \int_0^\theta \psi_P(\eta) d\eta = a_P \tau_P^2 \left[\frac{\theta^2}{2} - \psi_P(\theta) \right] \tag{48}$$

and

$$\beta(\theta) = a_E \tau_E \tau_P \int_0^\theta \psi_E(\eta) d\eta = a_E \tau_E^2 \left[\left(\frac{\tau_P}{\tau_E} \right)^2 \frac{\theta^2}{2} - \psi_E(\theta) \right] \tag{49}$$

In order to verify that the candidate solution in Eqs. (45)-(49) is

indeed optimal, sufficiency conditions have also to be satisfied. In previous studies [9, 10] it was proven that the existence of smooth isocost surfaces ("tubes") generated by the candidate solution, guarantees saddle point optimality.

Analysis of the present solution has shown [39], that in some subregions of the game space the isocost surfaces are only piece-wise smooth. A detailed analysis of such singular phenomena, induced by the eliptic vectograms [40], is out of the scope of the present discussion. Moreover, the singularity can be avoided by a slight modification of the admissible control set. Let us replace Eqs. (25) and (26) by

$$\hat{u} = \{u : u^T u \leq \hat{a}_P^2\}$$
$$\hat{v} = \{u : v^T v < \hat{a}_E^2\} \tag{50}$$

with

$$\hat{a}_P = a_P \cos \chi_P(0) \geq \hat{a}_E = a_E \tag{51}$$

resulting in circular vectograms for both players. This formulation is an exact one for "head on" or "tail chase" engagements, where $\cos \chi_P(0) = 1$, and represents a slightly pessimistic assumption for the pursuer for other initial geometries. Adopting this modification, the solution of the game becomes

$$\hat{p}^*(y,\theta) = -\hat{a}_P \xi^*$$
$$\hat{e}^*(y,\theta) = -a_E \xi^* \tag{52}$$

with

$$\hat{\xi}^* = \frac{y(\theta)}{\|y(\theta)\|} \quad \forall\ \theta \geq 0 \tag{53}$$

and

$$\hat{J}^*(y,\theta) = \|y\| - \hat{\alpha}(\theta) + \beta(\theta) \tag{54}$$

The isocost surfaces of the game have circular cross sections for all θ and they are smooth unless they intersect the $\theta(y=0)$ axis. This observation leads to define a "minimal tube," [10, 37, 39]. This is the isocost surface of $\hat{J}^*(y,\theta) = c_m$, which is tangent to $y = 0$. The point of tangency $\theta = \theta_s$ can be determined by solving

$$\frac{d\|y\|}{d\theta} = \hat{a}_P \tau_P^2\ \psi_P(\theta) - a_E \tau_E \tau_P\ \psi_E(\theta) = 0 \tag{55}$$

which leads to

$$\hat{a}_P \tau_P\ \psi_P(\theta_s) = a_E \tau_E\ \psi_E(\theta_s) \tag{56}$$

or more explicitly

$$\theta_s = \frac{\hat{a}_P}{a_E}\psi_P(\theta_s) + \frac{\tau_E}{\tau_P}\left(1-\exp\left[\frac{\tau_P}{\tau_E}\ \theta_s\right]\right) \tag{57}$$

This transcendental equation has a positive root $(\theta_s > 0)$ only if

$$\frac{\hat{a}_P}{a_E} < \frac{\tau_P}{\tau_E} \tag{58}$$

If this inequality is satisfied, there exists a "minimal tube" of

$\hat{J}_m^*(y,\theta) = c_m$ with

$$c_m = a_E \tau_P^2 \left[\frac{\hat{a}_P}{a_E} \psi_P(\theta_s) \left(1 - \frac{\tau_E}{\tau_P}\right) - \frac{\theta_s^2}{2}\left(\frac{\hat{a}_P}{a_E} - 1\right)\right] \tag{59}$$

A cross section of such "minimal tube" is shown in Fig. 5. The values of θ_s and $\tilde{c}_m \triangleq c_m/a_E\tau_P^2$ are depicted in Figs. 6 and 7 as the function of the maneuver ratio $\hat{a}_E/\hat{a}_P$ and the time constant ratio τ_E/τ_P .

If inequality (58) is satisfied, the (y,θ) game space can be decomposed to $\mathcal{D}^0$ and $\mathcal{D}^1$ by defining:

$\mathcal{D}^0 \triangleq$ the interior of the "minimal tube" for $\theta > \theta_s$ and
$\mathcal{D}^1$ as its complement.

The solution of the game (the optimal strategy pair and the Value) are given in $\mathcal{D}^1$ by Eqs. (52)-(54). In $\mathcal{D}^0$ any arbitrary admissible strategy pair is optimal and the Value is constant, given by (59).

If inequality (58) is not satisfied, point capture can be guaranteed by the pursuer in a part of the state space. For such case the decomposition of the state space is slightly modified by defining $\mathcal{D}^{0'}$ as

$$\mathcal{D}^{0'} \triangleq \{(y,\theta) \; : \; \|y(\theta)\| < \|y_s(\theta)\| = \hat{\alpha}(\theta) - \beta(\theta)\} \tag{60}$$

The optimal strategies in this domain are arbitrary and the Value is zero. Outside this region $(\|y(\theta)\| \geq \|y_s(\theta)\|)$ the optimal solution, expressed by Eqs. (52)-(54), remains valid.

C. Implementation and Validity

The closed form solution of the missile vs. aircraft game obtained in the previous subsection is based on perfect information (Ass. No. 8) implying that all state variables can be instantaneously and accurately measured by both players. Validity of this assumption has a major impact on the implementation of the optimal strategies.

The components of the "predicted miss distance" vector y in Eq. (38) are composed of 3 parts of different origin. The first two terms form the respective components of the "zero effort miss," proportional to the turning rate of the line of sight, $\dot{\sigma}$.

$$\begin{aligned} Y + \tau_P \theta \dot{Y} &= V_c (\tau_P \theta)^2 \dot{\sigma}_y \\ Z + \tau_P \theta \dot{Z} &= V_c (\tau_P \theta)^2 \dot{\sigma}_z \end{aligned} \tag{61}$$

V_c being the closing speed of the interception defined by (see Fig.2),

$$V_c = V_P \cos \chi_P(0) - V_E \cos \chi_E(0) = \text{const.} \tag{62}$$

The other two terms of Eq. (38) are the properly weighted lateral accelerations of the players. Acceleration can be easily measured onboard a flying vehicle but its estimation from another moving platform is a very complicated and tedious task. Based on this preliminary observation, the implementation of the analytical results of the previous subsection will be separately discussed for applications in optimal missile guidance and missile avoidance.

1. Optimal Missile Guidance

Measurements of the line of sight rate $\dot{\sigma}$, as well as the missile's own lateral accelerations has been of common practice in almost every guidance system. Estimation of target acceleration by Kalman filters has been also contemplated for some modern weapon systems in order to improve their performance. A recent study [43], however, indicated that such "optimal" guidance schemes are very sensitive to parameter variations and may not be cost effective.

Renouncing measurement (or estimation) of target acceleration by the pursuer is equivalent to assume instantaneous evader dynamics. Substituting $\tau_E = 0$ into Eqs. (57) and (58) yields [41]

$$\theta_{s_0} = \frac{\hat{a}_P}{a_E} \psi_P(\theta_{s_0}) \qquad (63)$$

$$c_{m_0} = a_E \tau_P^2 \left[\frac{\hat{a}_P}{a_E} \psi_P(\theta_{s_0}) - \frac{\theta_{s_0}^2}{2} \left(\frac{\hat{a}_P}{a_E} - 1\right)\right] \qquad (64)$$

These results represent the "worst case" analysis for the pursuer. The proposed strategies can serve, however, as directive for the optimal guidance law synthesis. The existence of a "minimal tube" and consequent decomposition of the (y, θ) state space are of major significance. The region $\mathcal{D}^0$, which is dominated by the pursuer, is characterized by small deviations from collision geometry and not too short pursuit times $(\theta > \theta_{s_0})$. Most trajectories start in this region (see Ass. No. 5) and for all of them

the predicted miss distance can be reduced to zero at $\theta = \theta_{s_0}$, against any admissible evasive maneuver, using an arbitrary admissible pursuer strategy. Thus, the optimal guidance law can be selected according to some other practical design considerations (as minimal control effort for example) and not by miss distance minimization. A time-varying linear feedback control law proposed in a previous 2-D study [22] can be an attractive choice. (Note that the computation of the "time to go," which requires measurements or estimation of range and range rate, is a relatively simple task for many guided missile systems.) The guaranteed miss distance c_m in Eq. (64) can be reduced by increasing missile maneuverability and decreasing its time of response. The final value of c_m shoul(serve as a guideline for warhead design.

2. Missile Avoidance

The conclusions of the analysis for missile avoidance are not encouraging. In cases which involve well designed missiles,inequality (58) is generally not satisfied. If the guidance system is capable of measuring or estimating the accelerations of the evader with adequate precision, zero miss distances are predicted for most initial conditions against all evasiv maneuvers. (In such cases aircraft survivability can be enhanced only by denying information from the missile).

Nevertheless, if it is known that the missile guidance law does not rely on estimation of evader acceleration, the analytical solution of IV B provides some clue for a practical evasion strategy. The proposed strategy

consists of a well timed "hard" maneuver in an optimal direction and does not require any elaborate measurement. According to the concept of the "minimal tube," maneuvering at $\theta > \theta_{s_0}$ in $\mathcal{D}^0$ does not contribute to the final outcome, because the pursuer can guarantee zero predicted miss distance at $\theta = \theta_{s_0}$. At that point $(y = 0, \theta = \theta_{s_0})$, however, evader maneuvers become effective. The optimal direction of the maneuver can be determined by inspection of Eq. (47) and the original elliptical vectograms in Fig. 4. It can be directly concluded that the miss distance is maximized by

$$\xi^{*T} = (0, \pm 1) \tag{65}$$

indicating that the optimal maneuver direction is perpendicular to the plane of collision (see Fig. 2) which is determined by the initial conditions.

The timing of this terminal maneuver need not be very accurate. It is sufficient to start the "hard" turn at some $\theta \geq \theta_{s_0}$ in order to guarantee a miss distance of c_{m_0} given by Eq. (64). In this sense, the proposed very simple avoidance strategy is optimal.

3. Validity of the Linearization

The main limitations of the analytical solution presented in IV B lie in the assumptions of perfect information and trajectory linearization. The implications of partial information were considered in the previous subsections dealing with implementation of the optimal strategies.

Other aspects of information imperfections are discussed in some detail in a recent paper [39]. In the following, the validity of trajectory linearization, which is essential to apply LDG techniques, is examined.

Linearization is based on the hypothesis (see II.A.5) that the directional changes during the engagement are not important. This assumption is valid [41] if two conditions are satisfied:

a. The direction change of the evader during the period of one time constant of the pursuer, defined also as the "dynamic similarity parameter" [44] of the pursuit-evasion problem

$$\tilde{\gamma} = \frac{a_E}{V_E} \tau_P \tag{66}$$

is small.

b. The optimal solution does not predict excessively long maneuvers i any direction.

The first condition can be observed before the linearization is adopte The second one, however, requires an "a posteriori" verification.

For trajectories starting outside the "minimal tube" the second condition is generally not fulfilled. In this region $(\mathcal{D}^1)$, constant direc tion maneuvers are optimal (see Eqs. (45) or (52)), leading to significant changes in interception geometry, unless the duration of the engagement is very short. For initial conditions in $\mathcal{D}^0$ the validity of the linearization depends on the actual strategy selected by the evader. Either a passi

evader behavior, based on minimum control effort consideration, or periodical evasive maneuvers [45] will maintain the initial geometry and consequently justify linearization. The short terminal maneuver initiated at $\theta = \theta_s$ (see IV.C.2) induces only minor direction changes.

It can be thus summarized that the linearized kinematic model provides a valid description of the missile vs. aircraft "end game" and engagements starting (and remaining) near to the initial collision course. For other initial conditions the original non-linear trajectory equations have to be solved.

V. SINGULAR PERTURBATION TECHNIQUE FOR NON-LINEAR PURSUIT-EVASION GAMES.

A. Preliminaries

The technique of singular perturbations (SPT) has been successfully used in approximate solutions of non-linear optimal control problems [25-27]. The first attempt to apply the same technique in non-linear zero-sum differential games is a very recent one [32]. In this section the basic principles of the method and its application for non-linear pursuit-evasion games are briefly summarized and illustrated by an example.

A dynamic system has a singularly perturbed structure if it involves a small parameter ε in such a way that the state vector $z \in R^n$ can be decomposed to subvectors $x \in R^m$ and $y \in R^{n-m}$ and state equation

$$\dot{z} = F(z,u,v,\varepsilon) \quad ; \quad z(t_0) = z_0 \tag{67}$$

can be written as

$$\dot{x} = f(x,y,u,v,\varepsilon) \quad ; \quad x(t_0) = x_0 \tag{68}$$

$$\varepsilon\dot{y} = g(x,y,u,v,\varepsilon) \quad ; \quad y(t_0) = y_0 \tag{69}$$

The existence of the singular perturbation parameter ε is always linked to the time scale separation of the state variables. If the functions $f(\cdot)$ and $g(\cdot)$ have the same order of magnitude it is clear that the rate of change of y is much faster, than the variations of x. There are however many non-linear dynamic systems of well known time scale separation between "fast" and "slow" variables, for which the direct identification of the small parameter is a complex task. In such problems (e.g.: aircraft performance optimization [25-26]) the singular perturbation parameter can be introduced artificially. The technique used in such "forced" singular perturbation (FSPT) problems is similar to the genuine SPT.

If the singular perturbation problem is "well posed", the solution of the reduced order system (where $\varepsilon = 0$) is a good approximation of the exact solution of the original problem, however, it generally cannot satisfy the end conditions of the fast variable. This discrepancy can be bridged by "boundary layer" solutions obtained using a stretched time scale

$$\tau = \frac{t-t_0}{\varepsilon} \tag{70}$$

For a uniformly valid approximation, the boundary layer solution must be asymptotically stable and should match the reduced order solution. If this condition is satisfied, a uniformly valid additive composite solution can be synthetized and serve as a zero-order approximation. If more accuracy is required, the variables can be expanded into asymptotic power series and higher order terms can also be taken to account.

The most attractive feature of SPT is that, if the original problem can be decomposed (assuming complete time scale separation) to successive boundary layers with a single active state variable in each, the solution is obtained in a feedback form. Moreover, in such a scheme genuine SPT and FSPT yield identical zero-order results.

Such feedback solutions have a great potential for real time airborne applications. The method of complete time separation is very effective for problems of initial boundary layers. Recent studies [46, 47] identified difficulties in obtaining feedback solutions for singular perturbation problems of terminal boundary layer. Structures of mathematical models for which SPT fails were also observed [48].

The application of SPT to differential games is more than a mere extension. Singularly perturbed linear differential games were investigated in the past [29 - 31], revealing the problem of "ill posedness". Though it has been shown [29], that singularly perturbed zero-sum linear

differential games are well posed, no such proof has been given for non-linear games. Application of SPT to non-linear zero-sum differential games is discussed in detail in a recent study [32]. In the following subsections the main results of the investigation are outlined and implemented for a non-linear pursuit-evasion game.

B. Singularly Perturbed Non-Linear Differential Game

1. Original Game

Consider a singularly perturbed autonomous non-linear dynamic system, controlled by two competing players (P, E), described by

$$\dot{x} = f(x,y,u,v,\varepsilon) \quad ; \quad x(t_0) = x_0 \tag{71}$$

$$\varepsilon\dot{y} = g(x,y,u,v,\varepsilon) \quad ; \quad y(t_0) = y_0 \tag{72}$$

ε being a small parameter, $x \in R^m$, $y \in R^{n-m}$, $x+y \in R^n$.

It is required to find an optimal strategy pair $p^*(\cdot,\varepsilon), e^*(\cdot,\varepsilon)$, selected from the set of admissible and playable pairs $p(\cdot,\varepsilon), e(\cdot,\varepsilon)$, such that

$$\begin{aligned} u(t,\varepsilon) &= p\{x(t),y(t),\varepsilon\} \qquad u \in U \\ v(t,\varepsilon) &= e\{x(t),y(t),\varepsilon\} \qquad v \in V \end{aligned} \tag{73}$$

transfers the system from the given initial conditions to a terminal

manifold

$$\psi[x(t_f),\varepsilon] = 0 \tag{74}$$

optimizing [P is the minimizer and E is the maximizer] the terminal cost function

$$J = G[x(t_f),\varepsilon] \tag{75}$$

Note that in this game both the terminal manifold and the cost depend only on the "slow" components of the state variable.

<u>Assumption 1</u>: The singularly perturbed differential game defined by Eqs.(71)-(75) has a saddle point solution characterized by the triplet $[p^*(\cdot,\varepsilon),e^*(\cdot,\varepsilon),J^*(\cdot,\varepsilon)]$ in a closed domain $\mathcal{D}$ of the state space R^n (J^* being of the class C^1).

2. <u>Reduced Game</u>

Let the reduced order game be defined by

$$\dot{x}^o = f(x^o,y^o,u^o,v^o,0) \quad ; \quad x(t_0) = x_0 \tag{76}$$

$$0 = g(x^o,y^o,u^o,v^o,0) \tag{77}$$

with $x^o \in R^m$. The terminal manifold is

$$\psi^o[x^o(t_f),0] = 0 \tag{78}$$

The cost function is given by

$$J = g[x^{o}(t_f), 0] \tag{79}$$

The set of admissible and playable strategy pairs for this game $p^{o}(\cdot, 0)$ and $e^{o}(\cdot, 0)$ are such that

$$\begin{aligned} u^{o}(t,0) &= p^{o}[x^{o}(t,0),0] \qquad u^{o} \in U \\ v^{o}(t,0) &= e^{o}[x^{o}(t,0),0] \qquad v^{o} \in V \end{aligned} \tag{80}$$

transfer the system to the terminal manifold.

<u>Assumption 2:</u> The reduced order game defined by Eqs.(76)-(80) has a saddle point solution, characterized by the triplet $[p^{*o}(\cdot,0), e^{*o}(\cdot,0)$ $J^{*o}(x^{o},0)]$ in a closed domain $\mathcal{D}^{r} \subset R^{m}$. (J^{*o} being C^{1}).

If both assumptions (1 and 2) hold, it can be asserted that for $\varepsilon \to 0$ optimal trajectories of both games approach each other everywhere, except for the fast variables near to t_0 . In other words:

For each point $x^{*o}(t,0)$ on the optimal trajectory of the reduced game, there exists a point $x^{*}(t,\varepsilon)$ on the projection of optimal trajectory of the original game to R^{m}, such that

$$x^{*}(t,\varepsilon) = x^{*o}(t,0) + O(\varepsilon) \qquad \forall\, t \in [t_0, t_f] \tag{81}$$

The fast variable in the reduced game is computed from Eq.(77)

$$y^{*o}(t,0) = \varphi[x^{*o}(t,0), u^{*o}(t,0) v^{*o}(t,0)] \tag{82}$$

and generally,

$$y^{*o}(t_0,0) \neq y_0 \tag{83}$$

As a consequence, a relation similar to Eq.(81)

$$y^*(t,\varepsilon) = y^{*o}(t,0) + O(\varepsilon) \tag{84}$$

can hold only on an interval not including t_0 . The disagreement of Eq.(83) can be overcame by introduction of an initial "boundary layer" game.

3. Boundary Layer Game

The zero order initial boundary layer game can be defined by the dynamics

$$\frac{dy_i}{d\tau} = g(x_0,y^i,u^i,v^i,0) \quad ; \quad y^i(0) = y_0 \tag{85}$$

where $y^i \in R^{n-m}$ and τ is the stretched time scale given in Eq.(70).

The cost function of the game is

$$J^i = \int_0^\infty \lambda_x^o(0)\, f[x_0,y^i(\tau,0)u^i(\tau,0)v^i(\tau,0)]\, d\tau \tag{86}$$

λ_x^o being the gradient of the optimal cost in the reduced game. The admissible strategy pairs are $p^i(\cdot,0)e^i(\cdot,0)$ such that

$$u^i(\tau,0) = p^i[x_0,y^i(\tau,0),0] \quad , \quad u^i \in U$$
$$v^i(\tau,0) = e^i[x_0,y^i(\tau,0),0] \quad , \quad v^i \in V \tag{87}$$

generate a solution of Eq.(85).

Playability of the boundary layer game is defined to guarantee asymptotic matching. A pair $[p^i(\cdot,0)e^i(\cdot,0)]$ is playable if it leads a trajectory starting at y_0 to the isolated equilibrium point $y^o(x_0)$ obtained from the solution of

$$g[x_0,y^o(x_0),u^o,v^o] = 0 \tag{88}$$

Moreover, the optimal trajectory has to satisfy

$$\lim_{\tau\to\infty} y^{*i}(\tau,0) = y^o(x_0) \tag{89}$$

Assumption 3: The boundary layer game has an optimal strategy pair $[p^{*i}(\cdot,0),e^{*i}(\cdot,0)]$ satisfying (89).

4. Composite Strategy Pair

Supposing that Assumptions 1, 2 and 3 are all satisfied, the following zero-order composite strategy pair is proposed as a candidate for the original singularly perturbed differential game:

$$\tilde{u}(t,0) = \tilde{p}[x,y,0]$$
$$\tilde{v}(t,0) = \tilde{e}[x,y,0] \tag{90}$$

such that the composite control functions $\tilde{u},\tilde{v}$ satisfy

$$\tilde{u}(t,0) = u^{*o}(t,0) + u^{*i}\left(\frac{t}{\varepsilon}, 0\right) - CP_u$$

$$\tilde{v}(t,0) = v^{*o}(t,0) + v^{*i}\left(\frac{t}{\varepsilon}, 0\right) - CP_v \tag{91}$$

for all $t \in [t_0,t_f]$.

CP_u and CP_v are the common parts of the reduced order and boundary layer controls cancelling out by the matching process.

The proposed strategy pair is obviously playable and can serve as a suboptimal approximation.

5. Extended Value

Let us define the outcome of the original game played with the composite strategy pair proposed in Eq.(90) as the Extended Value of the game.

$$J[x_0,y_0,\tilde{p}(\cdot,\varepsilon),\tilde{e}(\cdot,\varepsilon)] = G[\tilde{x}(\tilde{t}_f),\varepsilon] \triangleq \tilde{J}_\varepsilon(x_0,y_0\varepsilon) \tag{92}$$

The relationship between this suboptimal outcome and the exact optimal cost, i.e., the "Value" of the game $J^*(x_0,y_0,\varepsilon)$ is determined by the following theorem.

THEOREM 1. Suppose that Assumptions 1, 2 and 3, hold. Then the Extended Value of a singularly perturbed zero-sum differential game,

obtained by using the candidade strategy pair of Eq. (90), is bounded in both sides.

$$J^*(x_0, y_0, \varepsilon) - \psi_E(\varepsilon) \le \tilde{J}_\varepsilon(x_0, y_0, \varepsilon) \le J^*(x_0, y_0, \varepsilon) + \psi_P(\varepsilon) \tag{93}$$

$\psi_E(\varepsilon)$, and $\psi_P(\varepsilon)$ are correction terms which satisfy

$$\lim_{\varepsilon \to 0} \psi_E(\varepsilon) = \lim_{\varepsilon \to 0} \psi_P(\varepsilon) = 0 \tag{94}$$

This theorem, proven in Ref. [32], and illustrated in Fig. 8, has two direct consequences,

<u>COROLLARY 1</u>. The Extended Value of a singularly perturbed zero-sum differential game satisfies a weak saddle inequality expressed by

$$J(x_0, y_0 \tilde{p}, e, \varepsilon) - \psi_E(\varepsilon) \le \tilde{J}(x_0, y_0, \tilde{p}, \tilde{e}, \varepsilon) \triangleq \tilde{J}_\varepsilon(x_0, y_0, \varepsilon) \le J(x_0, y_0, p, \tilde{e}, \varepsilon) + \psi_P(\varepsilon) \tag{95}$$

which is a combination of Eqs. (93) and (16).

Substituting Eq. (94) into Eq. (93) leads to

<u>COROLLARY 2</u>. The Extended Value oı a singularly perturbed zero-sum differential game tends as a limit towards the Value of the game as ε approaches zero

$$\lim_{\varepsilon \to 0} \tilde{J}_\varepsilon(x_0, y_0, \varepsilon) = \lim_{\varepsilon \to 0} J^*(x_0, y_0, \varepsilon) \tag{96}$$

C. Application to Pursuit-Evasion Games

1. Game Characteristics

A class of frequently used pursuit-evasion games are characterized by the following:

a. There is a time scale separation between the variables describing the slow relative geometry and the fast variations of vehicle dynamics.

b. The dynamics of the slow variables are separately controlled by the players.

c. The dynamics of the fast variables are independent of the slow ones.

d. The fast variables are scalars and independently controlled by the players.

e. The terminal surface is defined by the slow variables only.

The dynamic equations of such game are

$$\dot{x} = f_P(x, y_P, u) + f_E(x, y_E, v) \quad , \quad x(t_0) = x_0 \tag{97}$$

$$\dot{y}_P = g_P(y_P, u) \quad , \quad y_P(t_0) = y_{P_0} \tag{98}$$

$$y_E = g_E(y_E, v) \quad , \quad y_E(t_0) = y_{E_0} \tag{99}$$

$$x \in R^{n-2} \quad , \quad y_P \in R^1 \quad , \quad y_E \in R^1 \quad , \quad u \in U \subset R^k \quad , \quad v \in V \subset R^\ell$$

Termination of the game (capture) is defined by

$$\psi[x(t_f)] = 0 \tag{100}$$

Let the pay-off of the game be the time of capture t_f .

The Hamiltonian of the game is

$$H = 1 + \lambda_x^T(f_P + f_E) + \lambda_P g_P + \lambda_E g_E \tag{101}$$

where λ_x, λ_P, λ_E are the respective gradients of the optimal cost determined by the adjoint equations

$$\dot{\lambda}_x = - \frac{\partial H}{\partial x} = - \lambda_x^T \left[\frac{\partial f_P}{\partial x} + \frac{\partial f_E}{\partial x}\right] \quad ; \quad \lambda_x(t_f) = \nu \text{ grad } \psi \qquad \nu > 0 \tag{102}$$

$$\dot{\lambda}_P = - \frac{\partial H}{\partial y_P} = - \lambda_x^T \frac{\partial f_P}{\partial f_P} - \lambda_P \frac{\partial g_P}{\partial y_P} \quad ; \quad \lambda_P(t_f) = 0 \tag{103}$$

$$\dot{\lambda}_E = - \frac{\partial H}{\partial y_E} = - \lambda_x^T \frac{\partial f_E}{\partial y_E} - \lambda_E \frac{\partial g_E}{\partial y_E} \quad ; \quad \lambda_E(t_f) = 0 \tag{104}$$

The optimal control function u*, v* have to satisfy

$$\min_{u \in U} \max_{v \in V} H = 0 \tag{105}$$

2. Transformation to a Singularly Perturbed Game

In many cases an appropriate transformation will lead to define a small parameter ε multiplying the left sides of Eqs.(98) and (99). If, however, the time scale separation is obvious, ε can be inserted artificially. We shall pursue this forced singular perturbation technique (FSPT), transforming Eqs.(97)-(99) and (102)-(104) to

$$\dot{x} = f_P(x,y_P,u) + f_E(x,y_E,v) \qquad x(t_0) = x_0 \tag{106}$$

$$\varepsilon\dot{y}_P = g_P(y_P,u) \qquad y_P(t_0) = y_{P_0} \tag{107}$$

$$\varepsilon\dot{y}_E = g_E(y_E,v) \qquad y_E(t_0) = y_{E_0} \tag{108}$$

and

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} \quad ; \quad \lambda_x(t_f) = \nu \operatorname{grad} \psi \qquad \nu > 0 \tag{109}$$

$$\varepsilon\dot{\lambda}_P = -\frac{\partial H}{\partial y_P} \quad ; \quad \lambda_P(t_f) = 0 \tag{110}$$

$$\varepsilon\dot{\lambda}_E = -\frac{\partial H}{\partial y_E} \quad ; \quad \lambda_E(t_f) = 0 \tag{111}$$

Equations (100) (101) and (105) do not change by the insertion of the perturbation parameter.

3. Reduced Game

Taking $\varepsilon = 0$, the equations of the reduced order game are obtained

$$\dot{x}^{o} = f_P(x^o, y_P^o, u^o) + f_E(x^o, y_P^o, v^o) \quad , \quad x^o(t_0) = x_0 \tag{112}$$

$$0 = g_P(y_P^o, u^o) \tag{113}$$

$$0 = g_E(y_E^o, u^o) \tag{114}$$

$$\dot{\lambda}_x^o = -\frac{\partial H}{\partial x^o} \quad ; \quad \lambda_x^o(t_f) = \nu \text{ grad } \psi, \ \nu > 0 \tag{115}$$

$$0 = -\frac{\partial H}{\partial y_P} \tag{116}$$

$$0 = -\frac{\partial H}{\partial y_E} \tag{117}$$

Inspection of Eqs.(113), (114), (116) and (117) indicates that y_P^o a y_E^o have become additional (but not independent) control variables of the reduced game.

Let the solution of the game yield the feedback controls $u^{*o}(x^o$ $v^{*o}(x^o)$ and consequently $y_P^o[u^{*o}(x^o)] = y_P^{*o}(x^o)$ and $y_E^{*o}[v^{*o}(x^o)] =$ $= y_E^{*o}(x^o)$. Generally these functions do not satisfy the initial conditions of the original game. For this purpose the solution of the boundary layer game is required.

4. Boundary Layer Game

Using the stretching transformation of Eq.(70) and taking $\varepsilon = 0$ the equations of this game are

$$\frac{dx^i}{d\tau} = 0 \Rightarrow x^i = \text{const} = x_0 \tag{118}$$

$$\frac{dy_P^i}{d\tau} = g_P(y_P^i, u^i) \quad ; \quad y_P^i(0) = y_{P_0} \tag{119}$$

$$\frac{dy_E^i}{d\tau} = g_E(y_E^i, v^i) \quad ; \quad y_E^i(0) = y_{E_0} \tag{120}$$

$$\frac{d\lambda_x^i}{d\tau} = 0 \Rightarrow \lambda_x^i = \text{const} = \lambda_x^o(x_0) = \lambda_{x_0}^o \tag{121}$$

$$\frac{d\lambda_P^i}{d\tau} = -\frac{\partial H}{\partial y_P^i} = -\lambda_x^T \frac{\partial f_P}{\partial y_P^i} - \lambda_P^i \frac{\partial g_P}{\partial y_P^i} \tag{122}$$

$$\frac{d\lambda_E^i}{d\tau} = -\frac{\partial H}{\partial y_E^i} = -\lambda_x^T \frac{\partial f_E}{\partial y_E^i} - \lambda_E^i \frac{\partial g_E}{\partial y_E} \tag{123}$$

It can be seen that since λ_x^o is constant, determined in the reduced game, the boundary layer equations of each player can be solved independently. From Eqs.(119) and (122) using Eq.(105) a feedback solution for u^{*i} is obtained

$$u^{*i} = u^{*i}(y_P^i, \lambda_{x_0}^o, x_0) \tag{124}$$

This solution has to satisfy also the condition of asymptotic stability

$$y_{P_0} + \int_0^\infty g_P(y_P^i, u^{*i})\,d\tau = y_P^{*o}(x_0) \tag{125}$$

Similarly, the optimal control of the evader is also given in a feed-back form

$$v^{*i} = v^{*i}(y_E^i, \lambda_{x_0}^o, x_0) \tag{126}$$

satisfying

$$y_{E_0} + \int_0^\infty g_E(y_E^i, v^{*i})\,d\tau = y_E^{*o}(x_0) \tag{127}$$

5. Composite Strategies

Let us consider the following candidate strategy pair as an approximation of the optimal pair of the game for all $t \in [t_0, t_f]$

$$\begin{aligned} \tilde{u}(t) &= \tilde{p}(x, y_P) \\ \tilde{v}(t) &= \tilde{e}(x, y_P) \end{aligned} \tag{128}$$

such that

$$\begin{aligned} \tilde{u}(t) &= u^{*o}[x(t)] + u^{*i}\left[x_0, y_P\left(\frac{t}{\varepsilon}\right)\right] - u^{*o}(x_0) = \tilde{u}(x, y_P) \\ \tilde{v}(t) &= v^{*o}[x(t)] + v^{*i}\left[x_0, y_E\left(\frac{t}{\varepsilon}\right)\right] - v^{*o}(x_0) = \tilde{v}(x, y_E) \end{aligned} \tag{129}$$

In this particular case the composite control is identical to the boundary layer control replacing the constant x_0 to the actual value of slow variable x.

$$\tilde{u}(x,y_P) = u^{*i}(x,y_P)$$
$$\tilde{v}(x,y_P) = v^{*i}(x,y_P) \tag{130}$$

Using this uniformly valid feedback strategy pair in the original game ($\varepsilon = 1$) it is observed that the termination of the independent boundary layers may not coincide. The pursuer's boundary layer terminates at $t = t_P$ determined by

$$y_{P_0} + \int_{t_0}^{t_P} g_P[y_P, \tilde{u}(x,y_P)]\,dt = y_P^o(x) \tag{131}$$

where $y_P^o(x)$ is the solution of

$$g_P[y_P^o(x), u^o(x)] = 0 \tag{132}$$

Similarly, t_E is given by

$$y_{E_0} + \int_{t_0}^{t_E} g_E[y_E, \tilde{v}(x,y_E)]\,dt = y_E^o(x) \tag{133}$$

with

$$g_E[y_E^o(x), v^o(x)] = 0 \tag{134}$$

The effectiveness of FSPT in non-linear pursuit-evasion games is demonstrated by the following simple example.

D. Example of Aircraft vs Aircraft Interception Game

1. Original Problem Formulation

The problem of interception of an unoffensive airplane by an offensive one, described in some detail in subsection III B 2, can be characterized by

(i) The initial range of separation is large enough to allow a "pure pursuit" phase.

(ii) The pursuer airplane is generally equipped with a missile of larger "capture range" than the turning radius of the airplanes. Consequently, the "end-game" phase disappears.

These features make this problem to be a specially suitable example for SPT application. For the sake of simplicity, a constant speed two-dimensional interception will be considered. The method, however, is equally applicable for variable speed and three-dimensional engagements.

The geometry of the pursuit in a plane is shown in Fig. 9 defining the four state variables $(R, \sigma, \chi_P, \chi_E)$.

The equations of motion are

$$\dot{R} = V_E \cos(\chi_E - \sigma) - V_P \cos(\chi_P - \sigma) \quad ; \quad R(t_0) = R_0 \tag{135}$$

$$\dot{\sigma} = \frac{1}{R}\left[V_E \sin(\chi_E - \sigma) - V_P \sin(\chi_E - \sigma)\right] \quad ; \quad \sigma(t_0) = \sigma_0 \tag{136}$$

$$\dot{\chi}_P = \omega_P u \quad ; \quad |u| \leq 1 \quad ; \quad \chi_P(t_0) = \chi_{P_0} \tag{137}$$

$$\dot{\chi}_E = \omega_E v \quad ; \quad |v| \leq 1 \quad ; \quad \chi_E(t_0) = \chi_{E_0} \tag{138}$$

ω_P , ω_E are the maximal turning rates of the pursuer and evader respectively.

The game terminates when the range between the players becomes equal to the capture radius ℓ

$$R(t_f) = \ell \quad ; \quad \dot{R}(t_f) < 0 \tag{139}$$

The pay-off to be optimized is the time of "capture" defined by Eq.(139)

$$J = t_f \tag{140}$$

Capture is guaranteed if $V_P > V_E$ and $a_P = \omega_P V_P \geq a_E = \omega_E V_E$. We assume that these conditions are satisfied. The Hamiltonian of the problem is

$$H = 1 + \lambda_R[V_E \cos(\chi_E-\sigma) - V_P \cos(\chi_P-\sigma)] +$$

$$+ \frac{\lambda_\sigma}{R} [V_E \sin(\chi_E-\sigma) - V_P \sin(\chi_P-\sigma)] + \lambda_P\omega_P u + \lambda_E\omega_E v \tag{141}$$

This problem belongs to the class of pursuit-evasion games analyzed in the previous subsection. Its exact solution was obtained by Simakova, more than a decade ago [49] and will be used for comparison. In Ref.[32] the system equations were transformed to a genuine singularly perturbed structure with

$$\varepsilon_P = \frac{V_P}{\omega_P R_0} \tag{142}$$

In this subsection the FSPT version will be solved demonstrating the equivalence of the two techniques for the zero-order approximation.

2. Forced Singular Perturbation Model

Since the rate of turn of the line of sight $\dot{\sigma}$ is much slower than the turning rates of the participating airplanes, χ_P and χ_E can be considered as "fast" variables.

The equations of this forced singularly perturbed dynamic system are

$$\dot{R} = V_E \cos(\chi_E-\sigma) - V_P \cos(\chi_P-\sigma) \qquad R(t_0) = R_0 \tag{143}$$

$$\dot{\sigma} = \frac{1}{R}[V_E \sin(\chi_E-\sigma) - V_P \sin(\chi_P-\sigma)] \qquad \sigma(t_0) = \sigma_0 \tag{144}$$

$$\varepsilon\dot{\chi}_P = \omega_P u \qquad \chi_P(t_0) = \chi_{P_0} \tag{145}$$

$$\varepsilon\dot{\chi}_E = \omega_E v \qquad \chi_E(t_0) = \chi_{E_0} \tag{146}$$

The set of the adjoint equations have the form

$$\dot{\lambda}_R = -\frac{\partial H}{\partial R} = \frac{\lambda_\sigma}{R^2}[V_E \sin(\chi_E-\sigma) - V_P \sin(\chi_P-\sigma)] \tag{147}$$

$$\dot{\lambda}_\sigma = -\frac{\partial H}{\partial \sigma} = \lambda_R [V_P \sin(\chi_P-\sigma) - V_E \sin(\chi_E-\sigma) +$$

$$+ \frac{\lambda_\sigma}{R}[V_E \cos(\chi_E-\sigma) - V_P \cos(\chi_P-\sigma)] \; ; \; \lambda_\sigma(t_f) = 0 \tag{148}$$

$$\varepsilon\dot{\lambda}_P = -\frac{\partial H}{\partial \chi_P} = -\lambda_R V_P \sin(\chi_P-\sigma) + \frac{\lambda_\sigma}{R} V_P \cos(\chi_P-\sigma) \quad ;$$

$$\lambda_P(t_f) = 0 \tag{149}$$

$$\varepsilon\dot{\lambda}_E = -\frac{\partial H}{\partial \chi_E} = \lambda_R V_E \sin(\chi_E-\sigma) - \frac{\lambda_\sigma}{R} V_E \cos(\chi_E-\sigma) \quad ;$$

$$\lambda_E(t_f) = 0 \tag{150}$$

The conditions of optimality which require $\underset{u}{\text{Min}}\ \underset{v}{\text{Max}}\ H = 0$

yield

$$u^* = -\,\text{sign}\ \lambda_P \qquad \lambda_P \neq 0 \tag{151}$$

$$v^* = \text{sign}\ \lambda_E \qquad \lambda_E \neq 0 \tag{152}$$

$$H^* = 0 \tag{153}$$

The missing condition for $\lambda_R(t_f)$ can be obtained from Eqs.(141) and (153)

$$\lambda_R(t_f) = 1/[V_P \cos(\chi_P-\sigma)_f - V_E \cos(\chi_E-\sigma)_f] \tag{154}$$

3. <u>Reduced Game</u>

In the reduced game (setting $\varepsilon = 0$) we have from Eqs.(145) and (146)

$$u^{*0} = 0 \tag{155}$$

$$v^{*0} = 0 \tag{156}$$

These results combined with the consequences of Eqs.(149) and (150) with $\varepsilon = 0$

$$\frac{\partial H}{\partial \chi_P^o} = 0 \tag{157}$$

$$\frac{\partial H}{\partial \chi_E^o} = 0 \tag{158}$$

indicate that in this auxiliary game the active controls are χ_P^o and χ_E^o. Eqs.(157) and (158) lead to

$$\text{tg}\ (\chi_P^o - \sigma^o) = \text{tg}\ (\chi_E^o - \sigma^o) = \frac{\lambda_\sigma^o}{R^o \lambda_R^o} = \text{tg}\ \alpha^o \tag{159}$$

resulting in

$$\chi_E^o = \chi_P^o = \alpha^o + \sigma^o \tag{160}$$

Substituting Eq.(160) into Eqs.(147) and (148) yields

$$\dot{\lambda}_R^o = \frac{\lambda_\sigma^o}{R^2} (V_E - V_P) \sin \alpha^o \tag{161}$$

$$\dot{\lambda}_\sigma^o = (V_E - V_P) \left[\lambda_R^o \sin \alpha^o - \frac{\lambda_\sigma^o}{R^o} \cos \alpha^o \right] \tag{162}$$

The brackets in Eq.(162) are zero due to Eq.(157) and (149), resulting in

$$\lambda_\sigma^o = \text{const} = \lambda_\sigma^o(t_f) = 0 \tag{163}$$

and consequently, using Eqs.(154) and (160)

$$\lambda_R^o = \text{const} = 1/(V_P - V_E) \tag{164}$$

Moreover, substitution of Eq.(163) into Eq.(159) determines

$$\alpha^{*o} = 0 \tag{165}$$

and the optimal control functions of the players are

$$\chi^{*o}_P = \chi^{*o}_E = \sigma^o(t) \tag{166}$$

Substitution of Eq.(166) into the equations of motion allow their integration

$$\sigma^o(t) = \text{const} \tag{167}$$

$$R^o(t) = R_0 - (V_P - V_E)\ (t - t_0) \tag{168}$$

4. Initial Boundary Layer Game

The set of equations for the initial boundary layer yields

$$\frac{dR^i}{d\tau} = 0 \rightarrow R^i = \text{const} = R_0 \tag{169}$$

$$\frac{d\sigma^i}{d\tau} = 0 \rightarrow \sigma^i = \text{const} \tag{170}$$

$$\frac{d\chi_P^i}{d\tau} = \omega_P u^i \ ; \qquad \chi_P^i(0) = \chi_{P_0} \tag{171}$$

$$\frac{d\chi_E^i}{d\tau} = \omega_E v^i \ ; \qquad \chi_E^i(0) = \chi_{E_0} \tag{172}$$

$$\frac{d\lambda_R^i}{d\tau} = 0 \Rightarrow \lambda_R^i = \text{const} \tag{173}$$

$$\frac{d\lambda_\sigma^i}{d\tau} = 0 \Rightarrow \lambda_\sigma^i = \text{const} \tag{174}$$

$$\frac{d\lambda_P^i}{d\tau} = - \frac{\partial H^i}{\partial \chi_P} = - \lambda_R^i V_P \sin (\chi_P^i - \sigma^i) + \frac{\lambda_\sigma^i}{R} V_P \cos (\chi_P^i - \sigma^i) \tag{175}$$

$$\frac{d\lambda_E^i}{d\tau} = - \frac{\partial H^i}{\partial \chi_E} = \lambda_R^i V_E \sin (\chi_E^i - \sigma^i) - \frac{\lambda_\sigma^i}{R} V_E \cos (\chi_E^i - \sigma^i) \tag{176}$$

The optimal control functions of the boundary layer are obtained from Eqs.(151) and (152)

$$u^{*i} = - \text{sign } \lambda_P^i \tag{177}$$

$$v^{*i} = \text{sign } \lambda_E^i \tag{178}$$

Matching of the constants of integration with reduced order game leads to

$$\sigma^i = \sigma^o \quad , \quad \lambda_\sigma^i = \lambda_\sigma^o \equiv 0 \quad , \quad \lambda_R^i = \lambda_R^o = \frac{1}{V_P - V_E} \tag{179}$$

Substitution of these constants and the optimal controls into Eqs.(171), (172), (175) and (176) results in two sets of independent equations, one for each player (as predicted previously).

The pursuer's equations are

$$\frac{d\chi_P^i}{d\tau} = - \omega_P \text{ sign } \lambda_P^i \; ; \qquad \chi_P^i(0) = \chi_{P_0} \tag{180}$$

$$\frac{d\lambda_P^i}{d\tau} = - \frac{V_P}{V_P-V_E} \sin (\chi_P^i-\sigma^o) \tag{181}$$

Similarly, the evader has

$$\frac{d\chi_E^i}{d\tau} = \omega_E \text{ sign } \lambda_E^i \;; \qquad \chi_E^i(0) = \chi_{E_0} \tag{182}$$

$$\frac{d\lambda_E^i}{d\tau} = \frac{V_E}{V_P-V_E} \sin (\chi_E^i-\sigma^o) \tag{183}$$

It is easy to see [32] that stable solutions require that

$$\text{sign } \lambda_P^i = \text{sign } (\chi_P^i-\sigma^o) \tag{184}$$

$$\text{sign } \lambda_E^i = - \text{ sign } (\chi_E^i-\sigma^o) \tag{185}$$

The asymptotically stable equilibrium points are (See Fig. 10)

$$\chi_{P_i}^i = \sigma^o \quad ; \quad \lambda_P^i = 0 \tag{186}$$

$$\chi_{E_i}^i = \sigma^o \quad ; \quad \lambda_E^i = 0 \tag{187}$$

These equilibria are reached in some finite time determined respectively by

$$\tau_P^* = |\sigma^o-\chi_{P_0}|/\omega_P \tag{188}$$

$$\tau_E^* = |\sigma^o-\chi_{E_0}|/\omega_E \tag{189}$$

After this time

$$\lambda_P^i \ (\tau \geq \tau_P^*) = 0 \tag{190}$$

$$\lambda_E^i \ (\tau \geq \tau_E^*) = 0 \tag{191}$$

satisfying the requirements of matching with the reduced game.

The optimal boundary layer controls can thus be expressed by substitution of Eqs.(184) and (185) into Eqs.(177) and (178) in a feedback form

$$u^{*i}(\tau) = u^*[\chi_P^i] = -\text{sign}\ (\chi_P^i - \sigma^o) \tag{192}$$

$$v^{*i}(\tau) = v^*[\chi_E^i] = -\text{sign}\ (\chi_E^i - \sigma^o) \tag{193}$$

5. Zero Order Composite Strategies

According to previous discussion in subsection V C, we propose the following uniformly valid strategy pair $\{\tilde{p}(\cdot),\tilde{e}(\cdot)\}$ as a suboptimal candidate for the original game

$$\tilde{p}\begin{cases} \tilde{u}(t) = -\ \text{sign}\ (\chi_P - \sigma) & \chi_P \neq \sigma \\ \tilde{u}(t) = \tilde{u}_s & \chi_P = \sigma \end{cases} \tag{194}$$

$$\tilde{e}\begin{cases} \tilde{v}(t) = -\ \text{sign}\ (\chi_E - \sigma) & \chi_E \neq \sigma \\ \tilde{v}(t) = \tilde{v}_s & \chi_E = \sigma \end{cases} \tag{195}$$

The singular controls $\tilde{u}_s$ and $\tilde{v}_s$ are such that

$$(\dot{\chi}_P)_s = (\dot{\chi}_E)_s = \dot{\sigma} \tag{196}$$

yielding

$$\tilde{u}_s = \frac{V_E}{R\omega_P} \sin (\chi_E - \sigma) \tag{197}$$

$$\tilde{v}_s = \frac{-V_P}{R\omega_E} \sin (\chi_P - \sigma) \tag{198}$$

This zero-order composite FSPT strategy, expressed by Eqs.(194)-(198) in a feedback form, is identical to the solution obtained using a genuine singular perturbation model [32].

This strategy consists for each player of three subarcs: (i) "hard" turn until the velocity vector is aligned with the line of sight; (ii) line of sight guidance, if the other player is still in phase (i); (iii) a straight line dash until capture. The resulting trajectories for different initial conditions are shown in Figs. 11 and 12.

6. Comparison to the Exact Solution

In the exact solution of Simakova [49], shown in broken lines in Figs. 11 and 12, the direction of the final dash is the common tangent of the players turning circles, determined by the initial conditions. The optimal strategy for each player is to align the velocity vector with this common tangent.

If the initial conditions are such that both players finish their hard turns simultaneously, the trajectories of the optimal and the SPT solutions, and consequently the outcomes of the game, are identical.

For any other initial conditions one of the players completes its "hard" turn earlier and following the suboptimal strategy of phase (ii) deviates from the optimal one. If it is the pursuer, the time of capture will be slightly longer than in the optimal game (see Fig. 11). If the evader reaches first the line of sight direction (see Fig. 12), the capture time of the SPT solution will be shorter than its value predicted by Simakova [49]. The differences are, however, very small. A quantitative comparison [32] has shown that for large initial ranges, relative to the radius of turn of the airplanes (ε_p in Eq. (142) less than 0.3), the differences are negligible (less than 1%).

The usefulness of the SPT can be appreciated by this excellent accuracy, in addition to the simple method of solution yielding feedback control laws. Moreover, the suboptimal SPT strategies can be easily implemented. They are based only on line of sight measurements (direction or rate) but do not require range information or the knowledge of the opponents flight direction. Note also that in this example the validity of Assumptions 1-3 can be directly verified.

VI CONCLUDING REMARKS

In this Chapter two analytical approaches, aimed to solve perfect information zero-sum differential games, were presented and applied to suitably formulated simple pursuit-evasion examples. It was shown that both methods — (i) a linear one (LDG) and (ii) a technique based on the concept of singular perturbations (SPT) — have a definite potential to yield closed form solutions for properly modelled "near real world" air combat problems. For sake of illustrative clarity the selected examples were of simplified nature (constant speed, first order time constant or, in one case, even two-dimensional geometry). However, the extension of these methods for more complex problems, such as variable speed, high order transfer functions, three-dimensional motion, currently under investigation [50], does not seem to present any difficulty.

It has to be admitted that each of the solution techniques exhibit some inherent limitation. A linear mathematical model cannot validly describe pursuit-evasion problems involving large changes of the interception geometry, which frequently occur in the initial phase. At the other end, the singular perturbation approach is unable to provide a "feedback" solution for engagements of rapidly varying terminal phase.

In order to solve pursuit-evasion problems, in which the initial acquisition and the "end-game" are of equal importance, an appropriate combination of both methods has to be investigated. It is hoped that the successful application of the individual solution techniques demonstrated in this Chapter will encourage such endeavour.

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REFERENCES

1. R. Isaacs, "Differential Games," John Wiley, New York, 1965.
2. R. Isaacs, "Games of Pursuit," The RAND Corp. Paper No. P-257, 1951.
3. N.K. Gupta, An Overview of Differential Games, in "Advances in Control and Dynamic Systems" (C.T. Leondes , ed.), Vol. 17, Academic Press, New York, 1980.
4. U.H.D. Lynch, Air Combat System Analysis Techniques, in "Advances in Control and Dynamic Systems (C.T. Leondes, ed.), Vol. 17, Academic Press, New York, 1980.
5. W.M. Getz and G. Leitmann, "Qualitative Differential Games with Two Targets," J. Math. Anal. Appl., 68, 421 (1979).
6. H.J. Kelley and L. Lefton, "A Preference Ordered Discrete-Gaming Approach to Air Combat Analysis," IEEE Trans. Auto. Cont. AC-23, 642 (1979).
7. D. Tabak, "Numerical Solutions of Differential Game Problems," Int. J. Sys. Sc., 6, 591 (1975).
8. J. Rader, Numerical Solutions for Zero-Sum Games, in "Advances in Control and Dynamic Systems," (C.T. Leondes, ed.) Vol. 17, Academic Press, New York, 1980.
9. A. Blaquière, F. Gerard, and G. Leitmann, "Quantitative and Qualitative Games," Academic Press, New York, 1969.
10. S. Gutman. "Remarks on Capture-Avoidance Games" (to appear in J. Optimiz. Theory Appl.).
11. J. Shinar, and D. Steinberg, "Analysis of Optimal Evasive Maneuvers Based on a Linearized Two-Dimensional Model," J. Aircraft, 14, 795 (1977).

12. J. Shinar, Y. Rotsztein, and E. Bezner, "Analysis of Three-Dimensional Optimal Evasion with Linearized Kinematics," J. Guidance Cont. 2, 353 (1979).

13. E. Cocayne, "Plane Pursuit with Curvature Constraints," SIAM J. Appl. Math. 15, 1511 (1967).

14. G.T. Rublein, "On Pursuit with Curvature Constraints," SIAM J. Cont., 10, 37 (1972).

15. U.H.D. Lynch, "Game Barriers and their Application in Air to Air Combat," Ph.D. Dissertation, Air Force Institute of Technology, Wright Patterson AFB, Ohio, 1973.

16. E.L. Miller, "Application of Differential Games to Air to Air Combat Problems," AFFDL-TR-7547, Air Force Flight Dynamics Laboratory, Wright Patterson AFB, Ohio, 1975.

17. Y.C. Ho, A.E. Bryson, Jr., and S. Baron, "Differential Games and Optimal Pursuit-Evasion Strategies," IEEE Trans. Auto. Cont. AC-10, 385 (1965).

18. C. Aumasson, "Approche du Problem de l'Interception Tri-dimensionnelle Optimale par la Theorie des Jeux Differentiels Linéaires Quadratiques, Recherche Aérospaciale, 176, 255 (1976)

19. M. Heymann, M. Pachter and R. Stern, "Max-Min Control Problems; A System Theoric Approach," IEEE Trans. Auto. Cont. AC-21, 455 (1976).

20. P, Bernhard, "Linear-Quadratic, Two-Person, Zero-Sum Differential Games: Necessary and Sufficient Conditions," J. Optimiz. Theory Appl., 27, 51 (1979).

21. S. Gutman, and G. Leitmann, "Optimal Strategies in the Neighborhood of a Collision Course," AIAA J., 14, 1210 (1976).

22. S. Gutman, "On Optimal Guidance for Homing Missiles," J. Guidance Cont., 2, 290 (1979).

23. A.W. Merz, "The Homicidal Chaufieur, A Differencial Game," Ph.D. Dissertation, Stanford University, Stanford, 1971.

24. J. Lewin, and G.J. Olsder, "Conic Surveillance Evasion," J. Optimiz. Theory Appl., 27, 107 (1979).

25. H.J. Kelley, Aircraft Maneuver Optimization by Reduced Order Approximations, in "Control and Dynamic Systems" (C.T. Leondes, ed.) Vol.10, Academic Press, New York, 1973.

26. M.D. Ardema, "Singular Perturbations in Flight Mechanics," NASA Report TM-X-62-380, 1974.

27. A.J. Calise, "Singular Perturbation Methods for Variational Problems in Aircraft Flight," IEEE Trans. Auto. Cont., AC-21, 343 (1976).

28. M.I. Freedman, and B. Granoff, "Formal Asymptotic Solution of a Singularly Perturbed Nonlinear Optimal Control Problem," J. Optimiz. Theory Appl., 19, 301 (1976).

29. B.F. Gardener, Jr., "Zero-Sum Nash Strategies for Systems with Fast and Slow Modes," Proc. 15th Allerton Conf. Communication Computers and Control, University of Illinois, Urbana, 96 (1977).

30. B.F. Gardener, Jr. and J.B. Cruz, Jr., "Well-Posedness of Singularly Perturbed Nash Games," J. Franklin Inst., 306, 355 (1978).

31. H.K. Khalil, and P.V. Kokotovic, "Feedback and Well-Posedness of Singularly Perturbed Nash Games," IEEE Trans. Auto. Cont. AC-24, 699 (1979).

32. N. Farber and J. Shinar, "An Approximate Solution of Singularly Perturbed Nonlinear Pursuit-Evasion Games," (to appear in J. Optimiz. Theory Appl.).

33. N.N. Krasovskii, "On the Problem of Pursuit in the Case of Linear Monotype Objects," PMM, 30, 263 (1966).

34. E.F. Miscenko, and L.S. Pontryagin, "Linear Differential Games," Sov. Math., 8 , 585 (1967).

35. P.N. Pshenichnyi, "Linear Differential Games," Auto. Telemekh. 1, 55 (1968).

36. H. Kimura, "Linear Differential Games with Terminal Pay-off," IEEE Trans. Auto. Cont., AC -15, 58 (1970).

37. W. Borgest, and P. Varaiya, "Target Function Approach to Linear Pursuit Problems," IEEE Trans. Auto. Cont., AC-16, 449 (1971).

38. T. Matsumoto, "A Class of Linear Evasion Problems," J. Optimiz. Theory Cont. 16 (1975).

39. J. Shinar, and S. Gutman, "Recent Advances in Optimal Pursuit and Evasion" (invited paper) Proc. 17th IEEE Conf. on Decis. Cont. San Diego, Calif., 960 (1979).

40. G.M. Anderson, "A Comparison of Air to Air Missile Guidance Laws Based on Optimal Control and Differential Game Theory," Proc. AIAA Guidance Cont. Conf., Boulder, Colorado, 306 (1979).

41. J. Shinar, and S. Gutman: "Three-Dimensional Optimal Pursuit and Evasion with Bounded Controls" (to appear in IEEE Trans. Auto. Cont.).

42. J. Shinar, S. Gutman, and M. Medinah, "On Singular Surfaces in a Linear Differential Game with Bounded Controls," TAE Report 387, Dept. Aero. Eng., Technion, Israel Inst. of Techn., Haifa, 1979.

43. F. Nesline, and P. Zarchan, "A New Look at Classical Versus Modern Homing Guidance," Proc. AIAA Guidance Cont. Conf. Boulder, Colorado, 230 (1979).

44. J. Shinar, and Y. Rotsztein, "Similarity Parameters in Pursuit-Evasion Problems," TAE Report 336, Dept. of Aero. Eng., Technion, Israel Inst. of Techn., Haifa, 1978.

45. E. Bezner, and J. Shinar, "Optimal Evasive Maneuvers in Conditions of Uncertainty," TAE Report 392 Dept. Aero. Eng. Technion, Israel Inst. of Tech., Haifa, 1979.

46. R.K. Mehra, R.B. Washburn, and S. Sajon, "A Study of the Application of Singular Perturbation Theory," Final Report on Contract NAS 1-15113, Scientific Systems Inc., Reading, Mass. 1979.

47. A. Merari, "On Singular Perturbation Technique Applied to Flight Mechanics Problems," M.Sc. Thesis, Technion, Israel Inst. of Techn., Haifa, 1980.

48. A. Merari, and J. Shinar, "Applicability of Forced Singular Perturbation Technique in Flight Mechanics," TAE Report 391. Dept. Aero. Eng., Technion, Israel Inst. of Techn., Haifa, 1980.

49. E.N. Simakova, "Differential Pursuit Game," Auto. Telemekh, 2, 5 (1967).

50. N. Farber, "Improved Dynamic Models for Air Combat Analysis," D.Sc. Dissertation, Technion, Israel Inst. of Techn., Haifa (in preparation).

LIST OF FIGURES

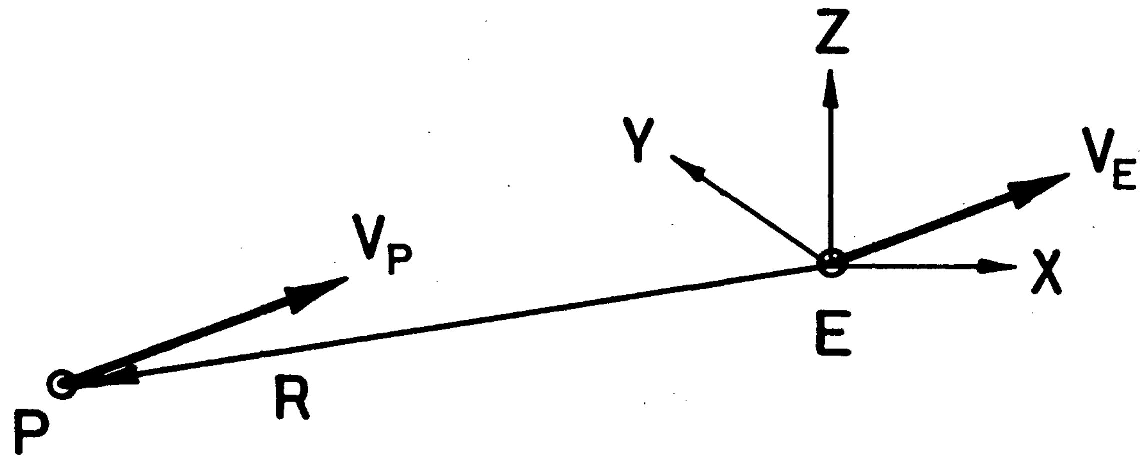

Fig. 1. Pursuit-Evasion Geometry.

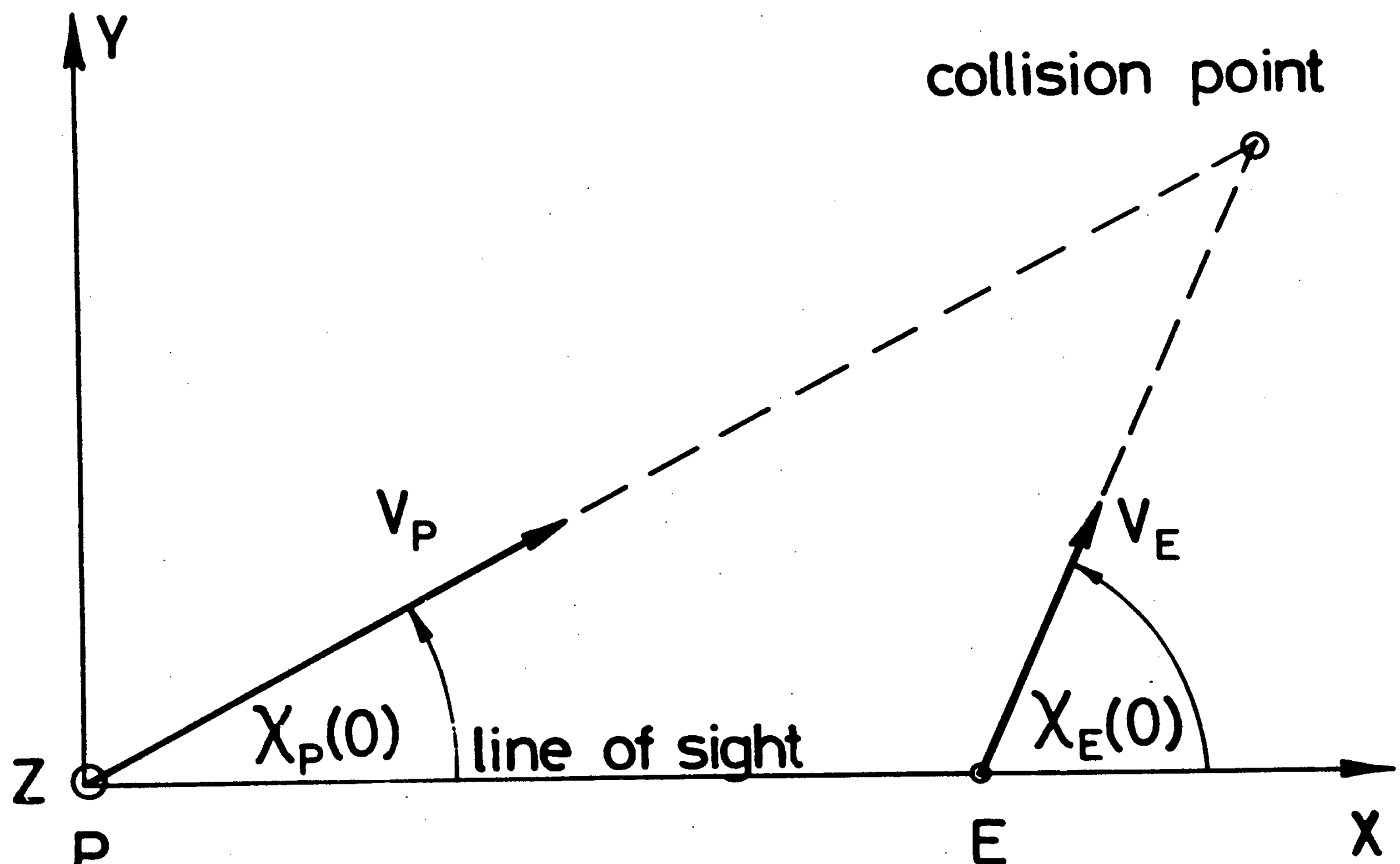

Fig. 2. Collision Course Geometry.

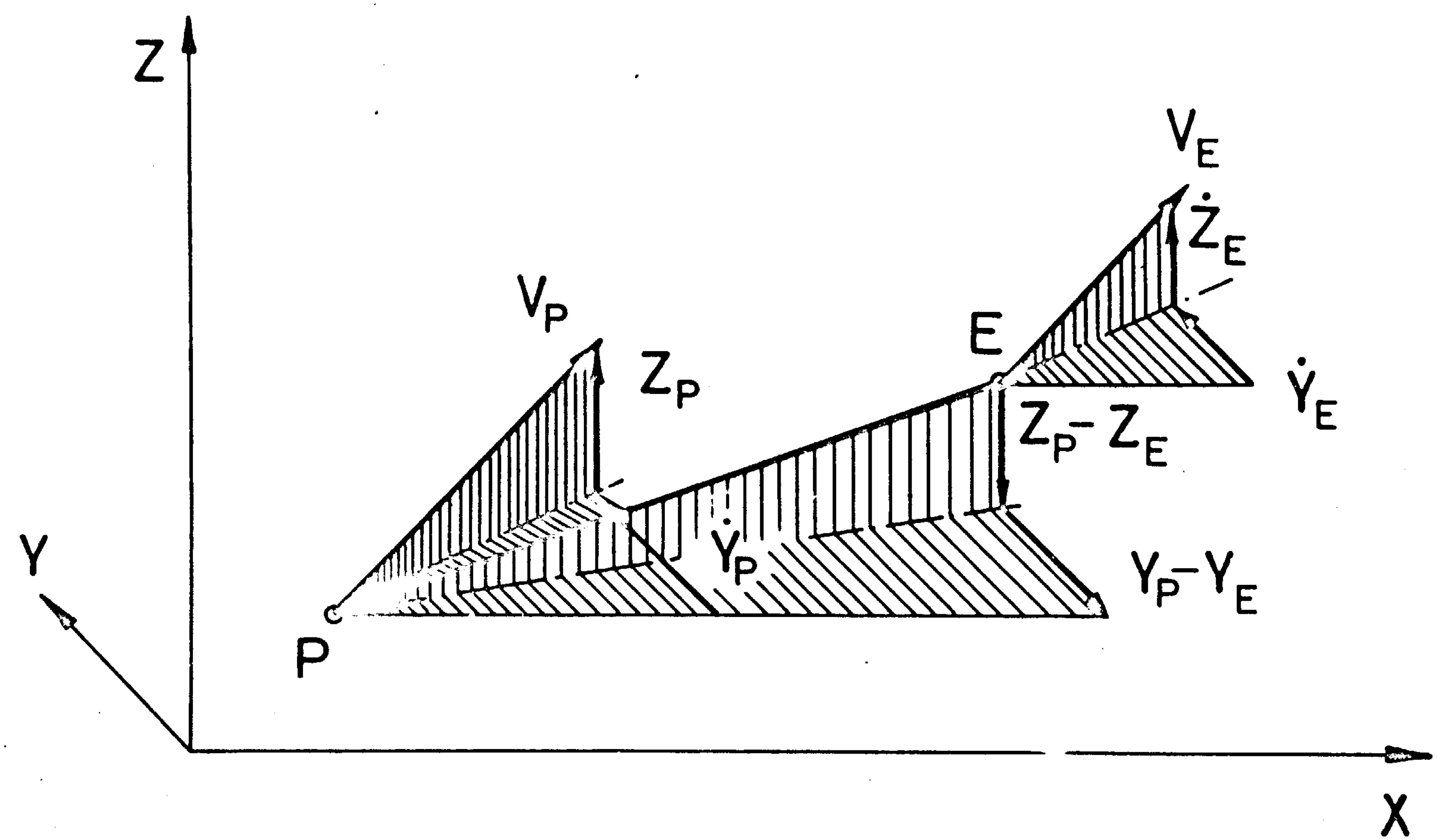

Fig. 3. Three-Dimensional Pursuit-Evasion.

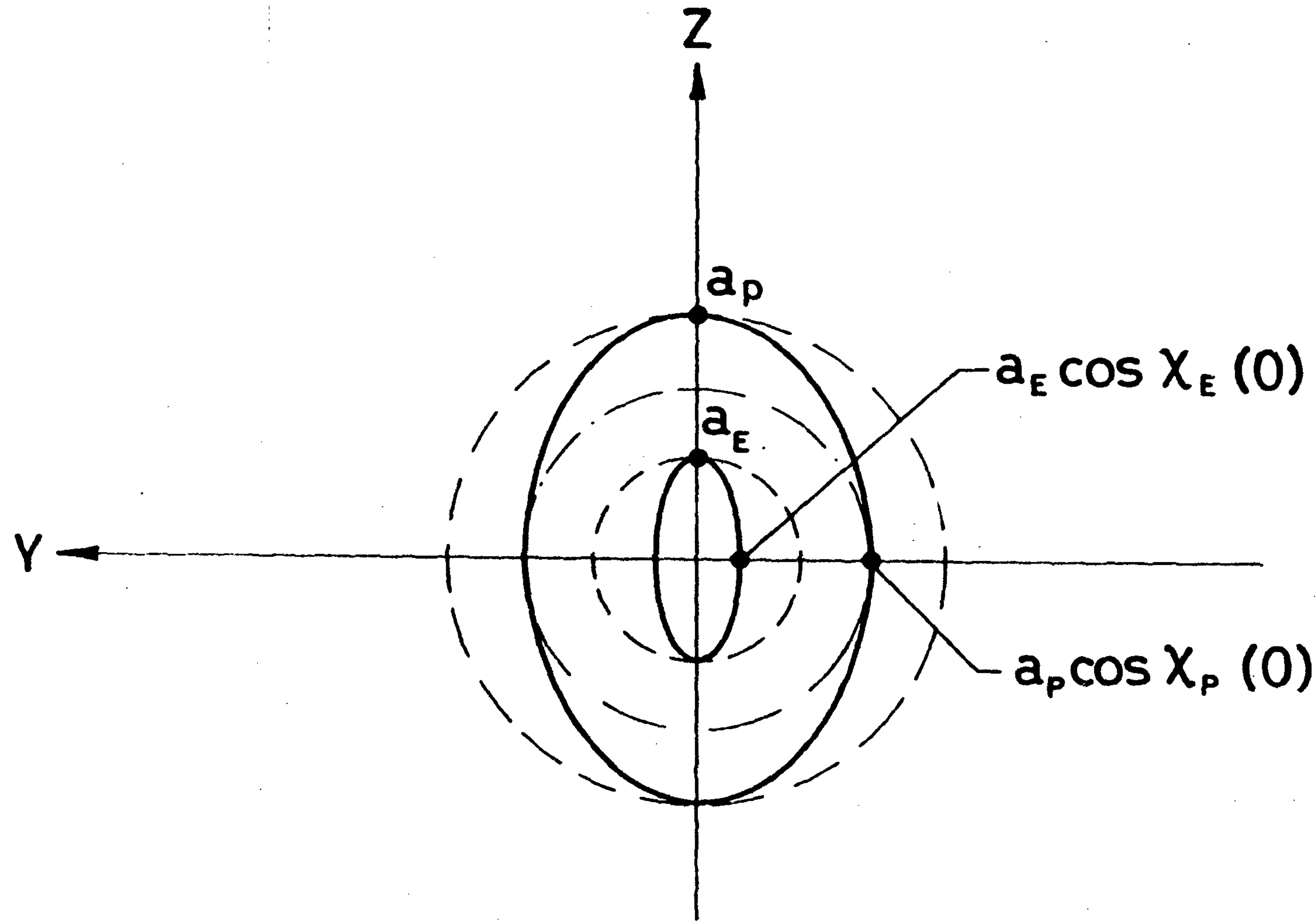

Fig. 4. Acceleration Vectograms Normal to the Initial Line of Sight.

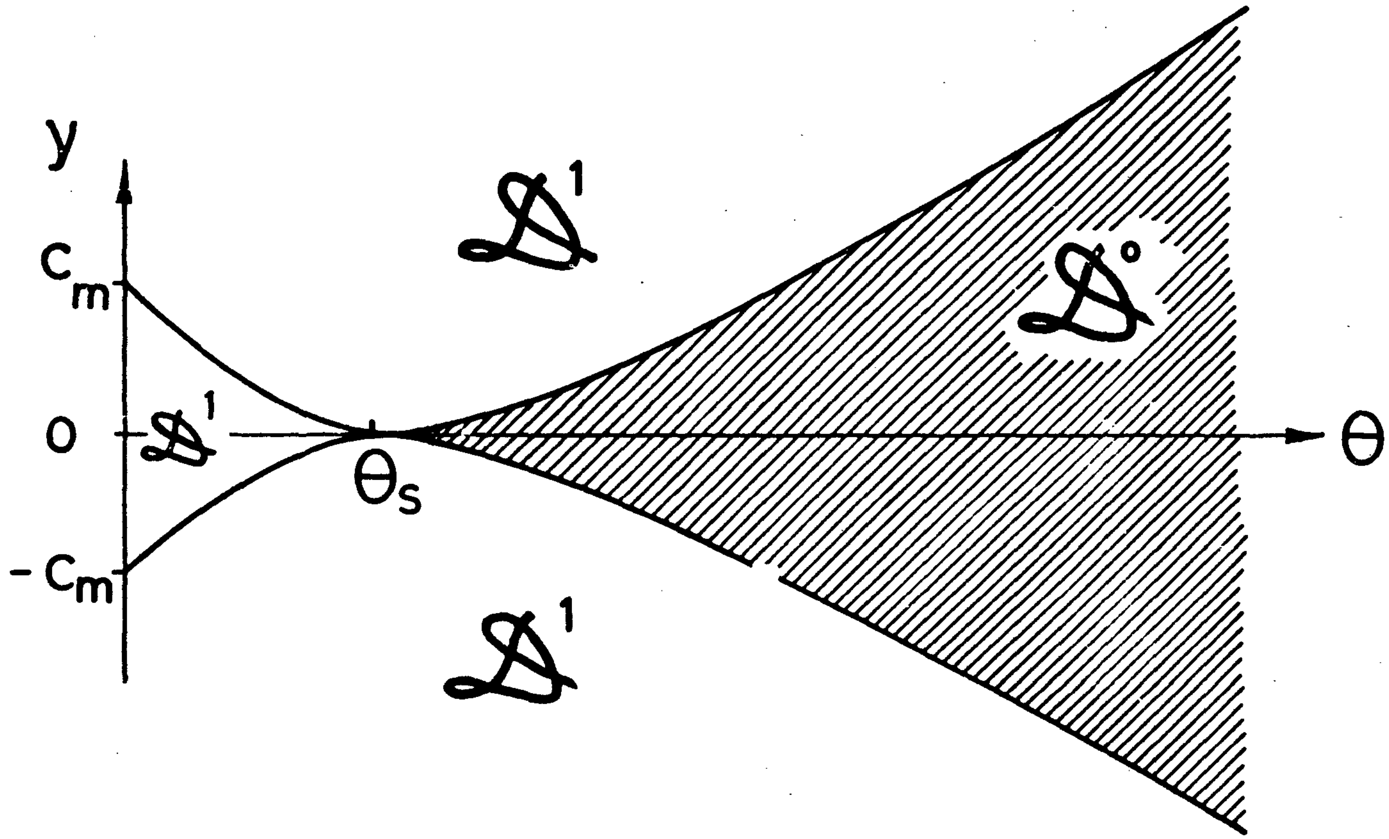

Fig. 5. Cross Section of the "Minimal Tube."

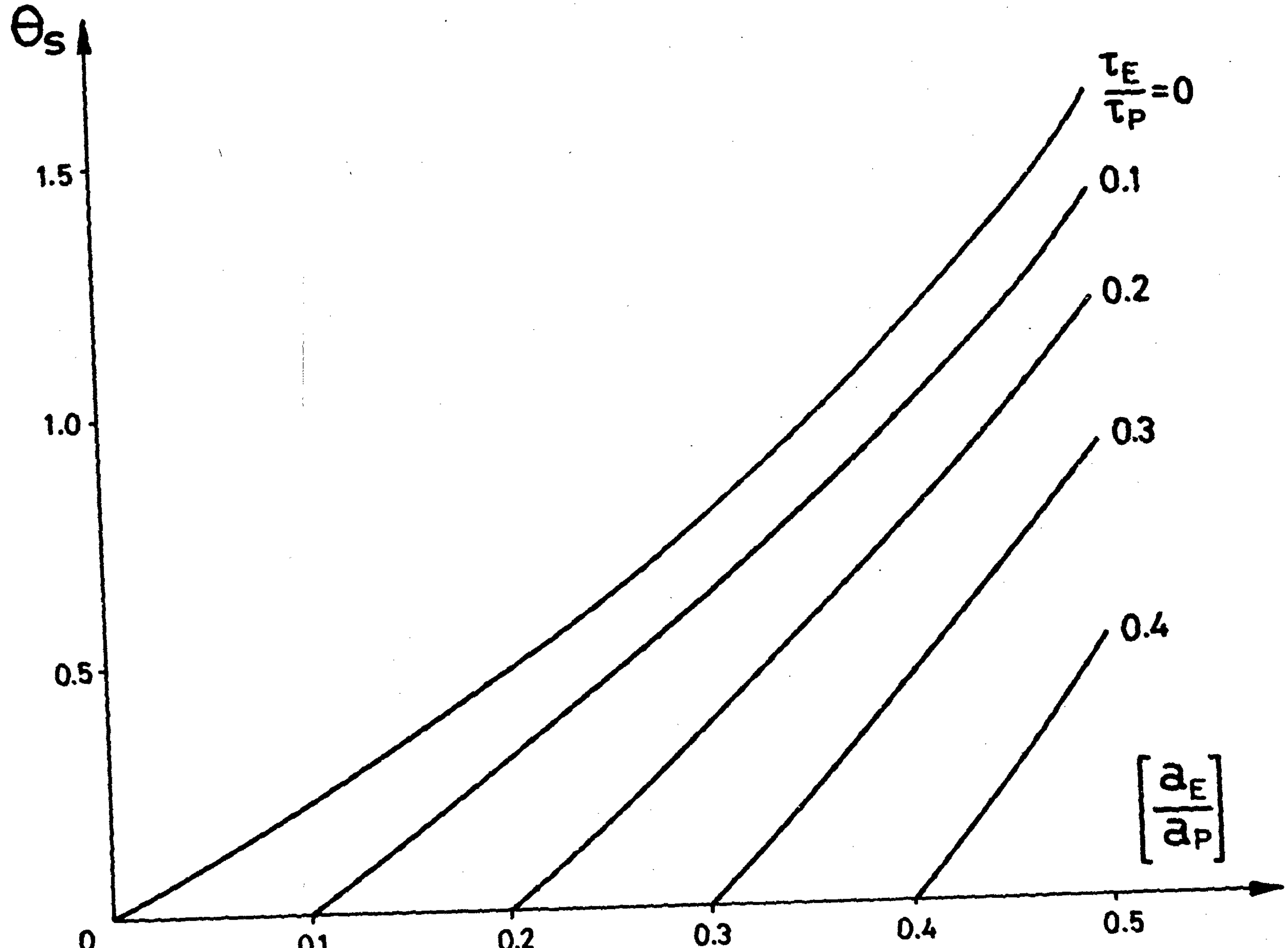

Fig. 6. Dependence of θ_s on the Maneuver and Time Constant Ratios.

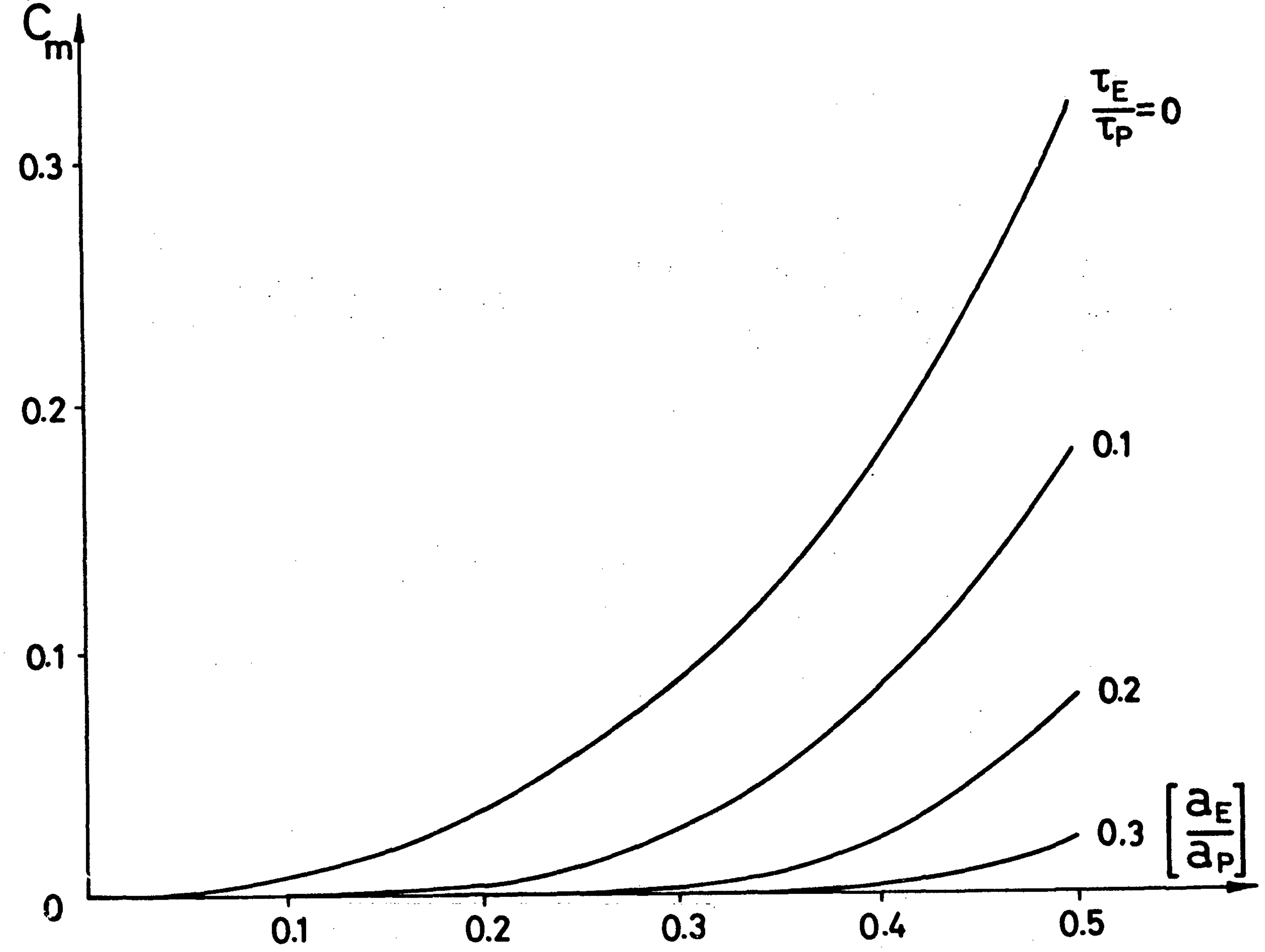

Fig. 7. Normalized Optimal Miss Distance.

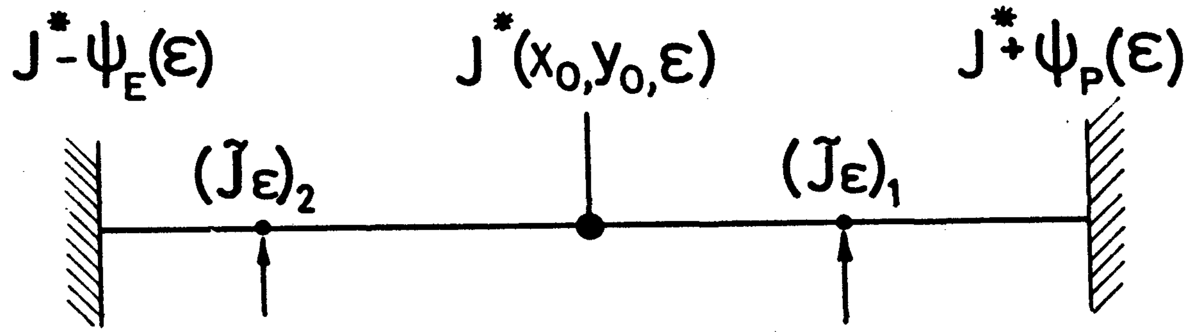

Fig. 8. Relation of the Extended Value to the Optimal Cost.

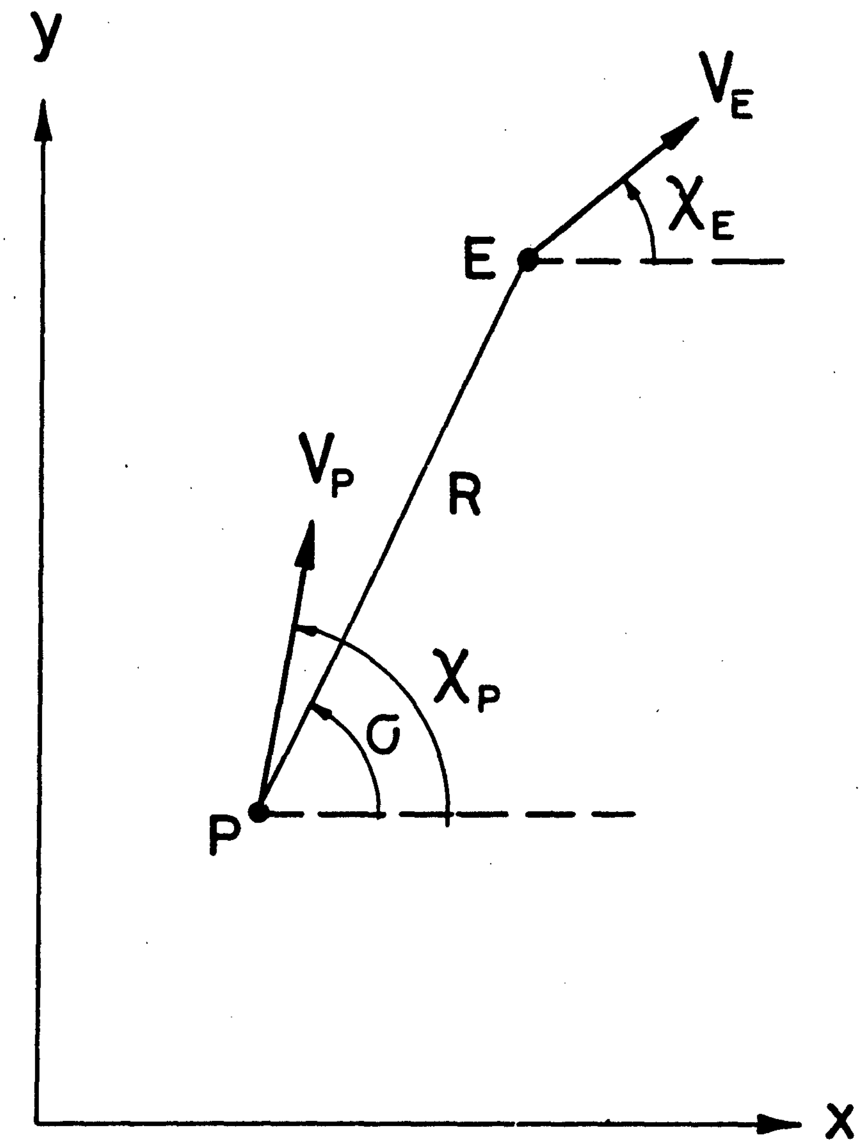

Fig. 9. Two-Dimensional Pursuit-Evasion.

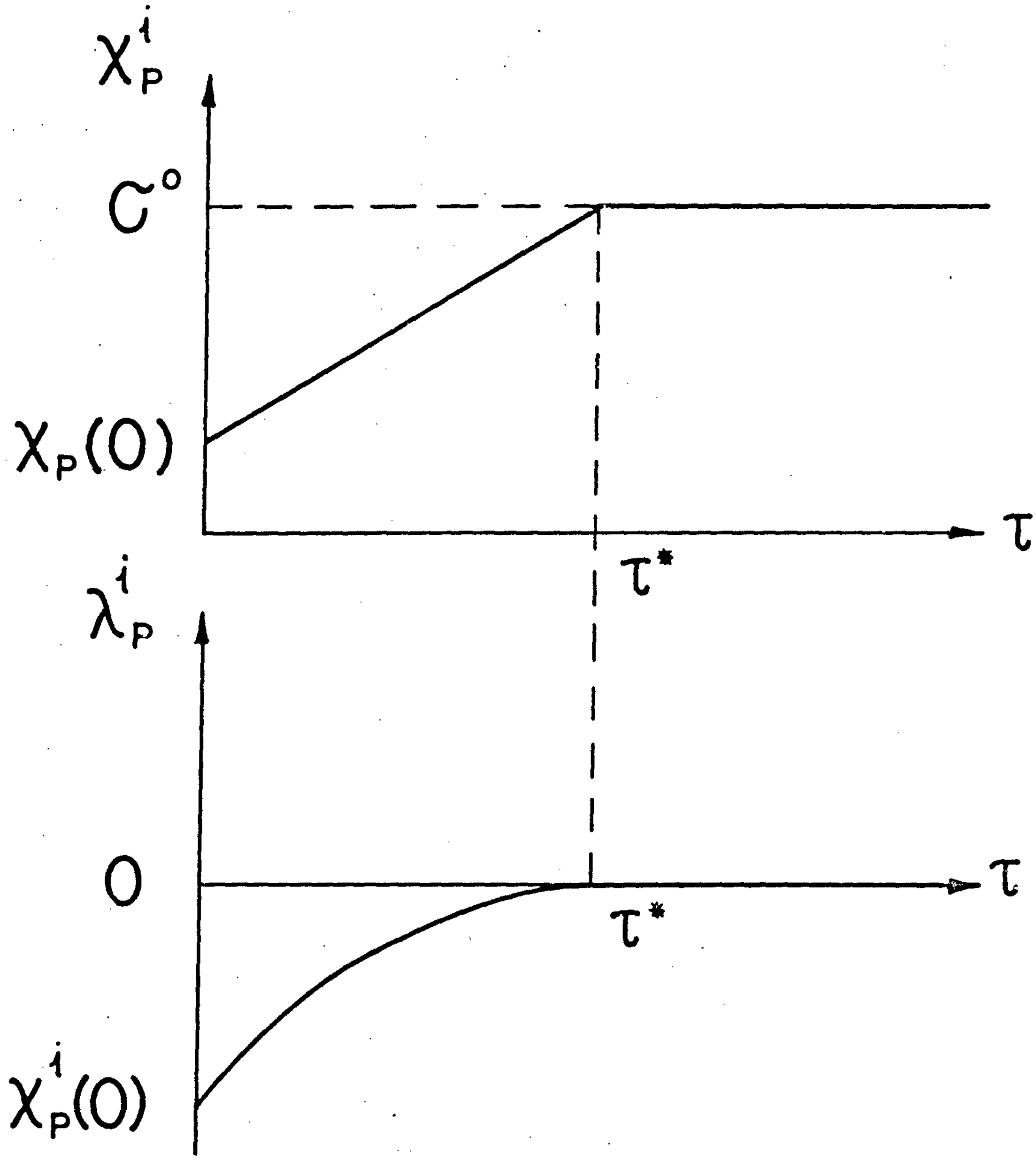

Fig. 10. Asymptotic Stability of the Boundary Layer Solution

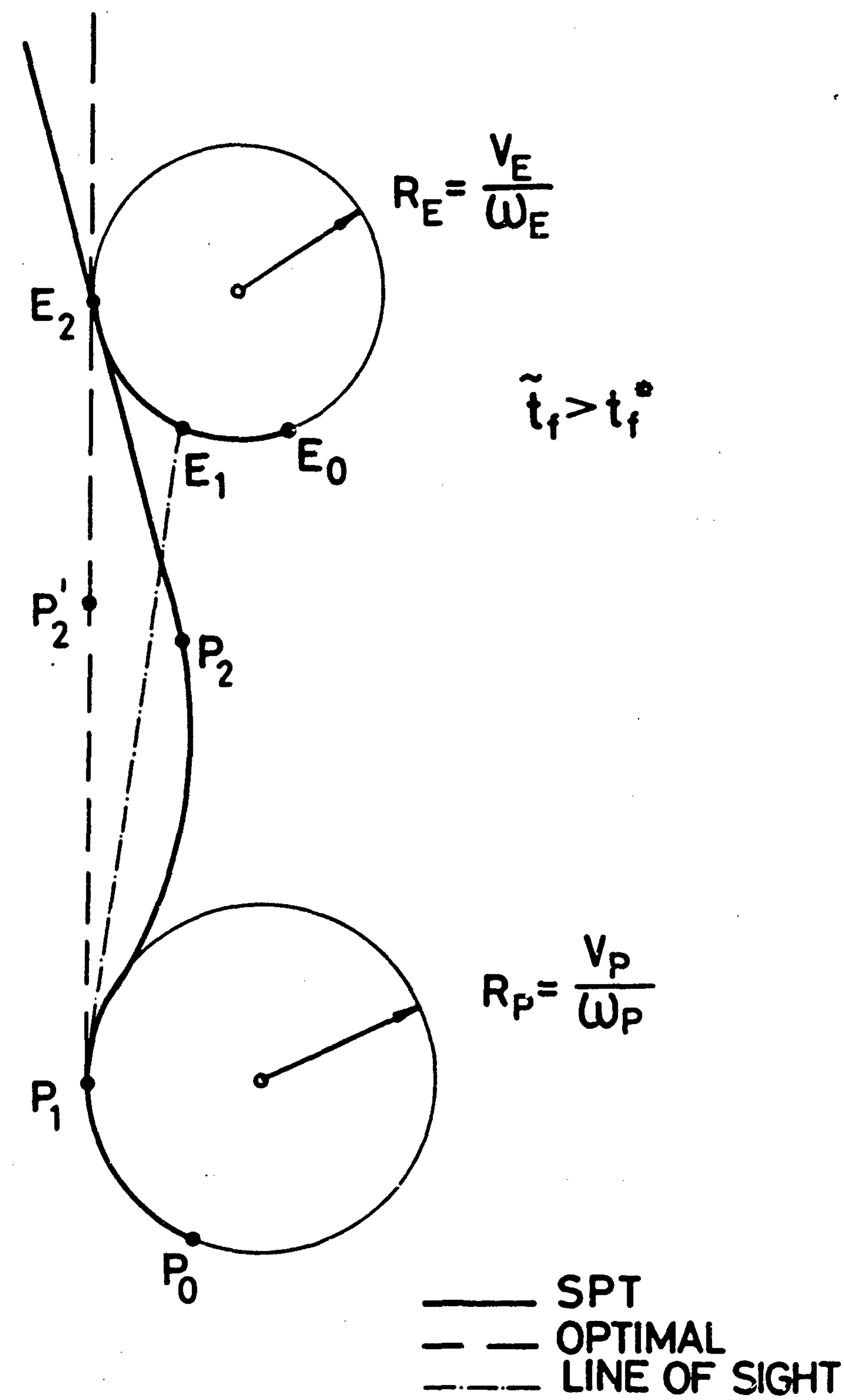

Fig. 11. Comparison of SPT Trajectory to the Exact Solution (Pursuer reaches line of sight first).

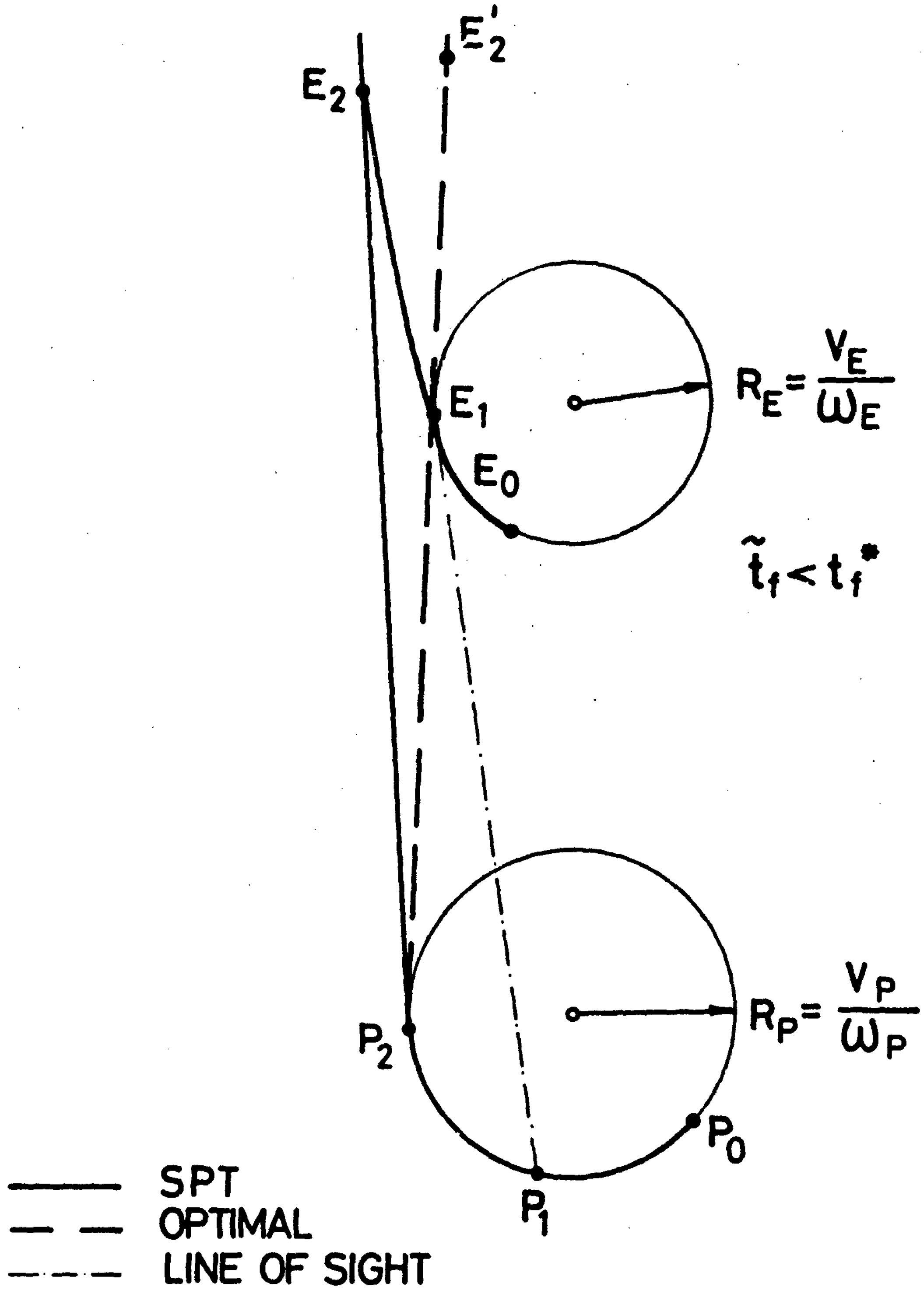

Fig. 12. Comparison of SPT Trajectory to the Exact Solution (Evader reaches line of sight first).